A shorter proof of the Graph Minor Algorithm – The Unique Linkage Theorem –*

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Abstract

At the core of the seminal Graph Minor Theory of Robertson and Seymour is a powerful theorem which describes the structure of graphs excluding a fixed minor. This result is used to prove Wagner's conjecture and provide a polynomial time algorithm for the disjoint paths problem when the number of the terminals is fixed (i.e, the Graph Minor Algorithm). However, both results require the full power of the Graph Minor Theory, i.e, the structure theorem.

In this paper, we show that this is not true in the latter case. Namely, we provide a new and much simpler proof of the correctness of the Graph Minor Algorithm. Specifically, we prove the "Unique Linkage Theorem" without using Graph Minors structure theorem. The new argument, in addition to being simpler, is much shorter, cutting the proof by at least 200 pages. We also give a new full proof of correctness of an algorithm for the well-known edge-disjoint paths problem when the number of the terminals is fixed, which is at most 25 pages long.

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1 Introduction

1.1 Graph Minors Algorithm

One of the deepest and most important bodies of work in graph theory is the Graph Minor Theory developed by Robertson and Seymour. At the heart of this theory is a theorem [33, Theorem 1.3] describing the structure of all graphs excluding a fixed graph as a minor. At a high level, the theorem says that every such graph can be decomposed into a collection of graphs each of which can "almost" be embedded into a bounded-genus surface, combined in a tree structure. Much of the Graph Minors series of articles is devoted to the proof of this structure theorem.

The main algorithmic result of the Graph Minor Theory is a polynomial-time algorithm for testing the existence of a fixed minor [31] which, combined with the proof of Wagner's Conjecture, implies the existence of a polynomial-time algorithm for deciding any minor-closed graph property. The existence of such a polynomial time algorithm has in turn been used to show the existence of polynomial-time algorithms for several graph problems, some of which were not previously known to be decidable [10]. It also leads to the framework of parameterized complexity developed by Downey and Fellows [8], which is perhaps one of the most active areas in the study of algorithms.

This algorithm is one of the most important polynomial time algorithms in theoretical computer science. The algorithm is relatively easy to describe. However, the proof of correctness of the algorithm (that is, the proof that the algorithm does in fact correctly determine the presence of a fixed graph as a minor) uses the full power of the Graph Minor Theory. More precisely, we can immediately reduce the problem to the case when the input graph has no large clique minor. However, the analysis of this case requires the development of the structure theorem. Our goal is to provide a new proof for the correctness of this algorithm that avoids many of the difficulties and technicalities in the original proof of Robertson and Seymour, and, specifically, avoids the use of the structure theorem.

The main purpose of this paper is to show the correctness of the Graph Minor Algorithm without using the structure theorem. This leads to a dramatically shorter and more simple proof of the correctness for the algorithm.

Much of the proof of the correctness of Graph Minor Algorithm in fact focuses on developing an algorithm for the disjoint paths problem. It will be more convenient for us, as well, to focus on the disjoint paths problem. We discuss this further in the next subsection.

1.2 The Graph Minors Algorithm vs. the k-disjoint paths problem

In the edge- (vertex-) disjoint paths problem, we are given a graph G and a set of k pairs of vertices, called *terminals*, in G, and we have to decide whether or not G has k edge- (vertex-) disjoint paths connecting given pairs of terminals. The problem of testing whether a given graph contains a fixed graph H as a minor can be trivially reduced to a bounded number of vertex disjoint path problems. Thus, a polynomial time algorithm for the k disjoint paths problem yields a polynomial time algorithm for minor testing, albeit with a worse runtime than that of the Graph Minors Algorithm. Moreover, the arguments for minor testing and the disjoint paths problem are analogous, although somewhat simpler to explain in the case of the disjoint paths problem. Finally, the k disjoint paths problem is also a classic problem the theory of algorithms, widely studied in its own right. For all these reasons, for the remainder of the article we will restrict our attention to the k disjoint paths problem. We will return only briefly to Graph Minors Algorithm to show how our results yield a short argument for the correctness of the Graph Minors Algorithm.

1.3 Background on the disjoint paths problem

The k disjoint paths problem, both in it's vertex and edge disjoint versions, is a central problem in algorithmic graph theory and combinatorial optimization. See the surveys [11, 36]. It has attracted attention in the contexts of transportation networks, VLSI layout and virtual circuit routing in high-speed networks or on the internet. A basic technical problem here is to interconnect certain prescribed "channels" on a chip such that wires belonging to different pins do not touch each other. In this simplest form, the problem mathematically amounts to finding disjoint trees in a graph or disjoint paths in a graph, each connecting a given set of vertices.

We now quickly look at previous results on the k disjoint paths problem. If k is as a part of the input of the problem, then this is one of Karp's original NP-complete problems [13], and it remains NP-complete even if G is restricted to be planar (Lynch [22]). The seminal work of Robertson and Seymour says that there exists a polynomial time algorithm (the actual runtime of the algorithm is $O(n^3)$. The time complexity is improved to $O(n^2)$ in [18]) for the disjoint paths problem when the number of terminals, k, is fixed. In the next subsection, we give an outline of this algorithm.

1.4 Robertson-Seymour Algorithm

In this subsection, we sketch Robertson and Seymour's algorithm for the k disjoint paths problem (see also [27]). At a high level, Robertson-Seymour's algorithm is based on the following two cases: either a given graph G has bounded tree-width (bounded by some function of k) or else it has large tree-width.

Case 1. Tree-width of G is bounded.

In this case, one can apply a dynamic programming argument to a tree-decomposition of bounded tree-width, see [1, 2, 31].

Case 2. Tree-width of G is large.

This second case again breaks into two cases depending on whether G has a large clique minor or not.

Case 2.1. G has a large clique minor.

If there exist disjoint paths from the terminals to this clique minor, then we can use this clique minor to link up the terminals in any desired way. Otherwise, there is a small cut set such that the large clique minor is cut off from the terminals by this cut set. In this case, we can prove that there is a vertex v in the clique minor which is *irrelevant*, i.e., the k disjoint paths problem is feasible in G if and only if it is also feasible in $G - v$. We then recursively apply the algorithm to $G - v$.

Case 2.2. G does not have a huge clique minor.

In this case, one can prove that, after deleting a bounded number of vertices, there is a huge subgraph which is essentially planar. Moreover, this huge planar subgraph itself has very large tree width. This makes it possible to prove that the middle vertex v of this wall is irrelevant. Again, we recursively apply the algorithm to $G - v$.

The analysis of Cases 1 and 2.1 is relatively easy. It is the analysis of Case 2.2 that gives rise to the majority of the work. The analysis of this case requires the whole series of graph minor papers and the structure theorem of [33].

1.5 Our main contributions – Unique linkage theorem

The analysis of Case 2 in the previous subsection can be reduced to the Unique Linkage Theorem without excessive difficulty. In fact, in the corresponding argument for the Graph Minors Algorithm, this is the only place in the proof of correctness which requires the full structure theorem. Before stating the theorem, we give some notation.

A *linkage* is a graph where every component is a path (possibly trivial). The *order* of the linkage is the number of components. In slightly sloppy notation, we will use $P \in \mathcal{P}$ to refer to a component P of the linkage $\mathcal{P}$. Two linkages $\mathcal{P}$ and $\mathcal{P}'$ are *equivalent* if they have the same order and for every component P of $\mathcal{P}$, there exists a component P' of $\mathcal{P}'$ such that P and P' have the same endpoints. We say a linkage $\mathcal{P}$ in a graph G is *unique* if for all linkages $\mathcal{P}'$ in G equivalent to $\mathcal{P}$, we have that $V(\mathcal{P}') = V(\mathcal{P})$.

We are now ready to state the theorem, which is the main result of Graph Minors XXI [34].

Theorem 1 (**The Unique Linkage Theorem [34]).** *For all $k \geq 1$, there exists a value $w(k)$ such that the following holds. Let $\mathcal{P}$ be a linkage of order k in a graph G with $V(G) = V(\mathcal{P})$. If $\mathcal{P}$ is unique, then the tree-width of G is at most $w(k)$.*

The current proof given by Robertson and Seymour [34] needs the full power of the graph minor structure theorem, but they predicted that there exists a simpler proof avoiding the use of the Graph Minor structure theorem. Our main contribution is to confirm that they are right– we provide such a short proof. In fact, our proof less than 25 pages, and gives rise to an explicit bound for the tree-width $w(k)$, while the original algorithm does not.

We now mention several consequences of our new shorter proof. First, we clarify how the unique linkage theorem implies that the vertex v in Case 2 is irrelevant. This was easy to prove for Case 2.1. The formal argument is given in Sections 5 and 6 in [31]. We are left with Case 2.2. The main result in [35] says that the existence of the irrelevant vertex in Case 2.2 can be reduced to the unique linkage problem. Let us observe that the arguments in [35] does NOT use the graph minor structure theorem. It is totally self-contained. Our proof of the Unique Linkage Theorem currently uses several tools from [29] for graphs embedded on surfaces of bounded genus (again, these tools do not depend on the structure theorem). Thus together with [35] and [29], our proof of the Unique Linkage Theorem provides a proof of the correctness of the k-disjoint paths algorithm which avoids the use of the graph minor structure theorem. At the moment, we believe that we also have a much shorter proof of the main result in [35] and the aspects of [29] which we use. This would lead to a correspondingly short, self-contained proof of the k-vertex disjoint paths algorithm.

Second, when we consider instead the k-edge disjoint paths problem, we are able to avoid the need for the work of [35]. This allows us to give a self-contained argument for the proof of correctness of the k-edge disjoint paths problem. We present the argument in the next subsection.

Finally, one of the original applications of the Unique Linkage Theorem is to verify the "intertwining conjecture" of Lovász [21] and Milgram and Unger [24]. The conjecture states that for every two graphs G_1 and G_2, there is a finite list $H_1, \ldots, H_n$ of graphs, such that G topologically contains both G_1 and G_2 if and only if it topologically contains one of $H_1, \ldots, H_n$ (G *topologically* contains H if some subgraph of G is isomorphic to a subdivision of H). A proof of this conjecture follows from the unique linkage theorem, as proved in [34]. But our proof, together with the proof in Section 11 of [34] gives rise to a short self-contained proof of this conjecture, which is, we believe, of independent interest. As pointed out in [34], our proof yields an algorithm that given two graphs G_1 and G_2, computes $H_1, \ldots, H_n$ above.

We conclude with a few words on possible applications of our new proof. Kernelization is a technique for creating algorithms for fixed-parameter tractable problems. This concept has attracted recent interest within the framework of parameterized complexity. See, for example, [3]. The approach is based on the observation that a problem is fixed-parameter tractable if and only if it is kernelizable and decidable. The idea of kernelization is to reduce the size of the input X to a function of k in polynomial time. When the input is bounded by k, we can use any exponential time algorithm, for example brute-force search, to find a solution of the problem. A basic technique in kernelization arguments is to find an "irrelevant" vertex for the problem, and reduce the size of the input. This is exactly what we do for the disjoint paths problem, hence we hope that our new short proof might yield new technical methods in this line of inquiry.

Algorithms for H-minor-free graphs for a fixed graph H have been studied extensively; see e.g. [4, 5, 6, 12]. In particular, it is generally believed that algorithms for planar graphs can often be generalized to H-minor-free graphs for any fixed H. Results from graph minors, and the unique linkage theorem in particular, are essential for these arguments. For example, the topological embedding algorithms given in [14, 15, 16, 17, 23] partially depend on the unique linkage theorem. Also, linear time algorithms for the disjoint paths problem when an input graph is planar [26] or an input graph is bounded genus [9, 20] heavily depend on the unique linkage theorem. Thus we anticipate that our new proof will have further applications along these lines.

1.6 The edge-disjoint paths problem

Using our new proof of Unique Linkage Theorem, we give a short proof of correctness for the *k-edge disjoint paths problem*.

Input: A graph G with n vertices and m edges, k pairs of vertices (s_i, t_i), called *terminals*, $i = 1, \dots, k$, in G.

Output : Edge-disjoint paths $P_1, P_2, \dots, P_k$ in G such that P_i joins s_i and t_i for $i = 1, 2, \dots, k$.

We assume that k is fixed. We will need the following definitions. For a vertex set X in a graph $G = (V, E)$, let $\delta(X)$ be the set of edges between X and $V \setminus X$. For a graph $G = (V, E)$, its *line graph* $L(G)$ is the graph whose vertex set is E such that two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G. To simplify the description, when we consider the line graph of a graph with terminals, we assume that exactly one edge is incident to each terminal by adding a new terminal and a single edge to G. Let $\tilde{s}_1, \dots, \tilde{s}_k, \tilde{t}_1, \dots, \tilde{t}_k$ be the edges incident with the terminals $s_1, \dots, s_k, t_1, \dots, t_k$ in G, respectively. Then, one can see that the edge-disjoint paths problem in G with respect to the terminals $s_1, \dots, s_k, t_1, \dots, t_k$ is equivalent to the vertex-disjoint paths problem in $L(G)$ with respect to the terminals $\tilde{s}_1, \dots, \tilde{s}_k, \tilde{t}_1, \dots, \tilde{t}_k$.

As proved in [19], an instance of the k edge-disjoint paths problem can be reduced to an instance satisfying the following conditions:

(R1) All vertices have degree at most $2k - 1$.

(R2) G has no vertex set X such that $|X| \geq 2$, X contains no terminals, and $|\delta(X)| \leq 3$.

(R3) G and $L(G)$ has no clique minor of size $3k$.

We call these operations *simple reductions*. Although it is easy to find a vertex of high degree (as in (R1)) and a ≤ 3-edge-cut in a given graph (as in (R2)), it is not easy to find a large clique minor.

The following theorem, which is the main result in [19], characterizes the instances of the edge-disjoint paths problem, and shows a way to find a large clique minor.

Theorem 2. *For any instance of the k-edge-disjoint paths problem and for any integer $h \geq 2$, there exists an integer $f(k, h)$ such that one of the following holds:*

(A) The instance violates at least one of (R1), (R2), and (R3). That is, one of the simple reductions can be applied to the instance.

(B) The input graph has tree-width at most $f(k, h)$.

(C) The input graph contains a wall W of size h with the outer face boundary C with the following properties:

(a) $G - C$ consists of two parts X and Y such that $X \cup C$ contains the whole wall W.

(b) Every vertex of $X \cup C$ has degree at most three.

(c) $X \cup C$ does not contain any terminal.

(d) $X \cup C$ can be embedded in a plane with the outer face boundary C.

We can find one of (A), (B) and (C) in $O(m)$ time. Furthermore, if the instance satisfies (R1) and (R2), but does not satisfy (B) and (C), then we can find a clique minor of size $3k$ in G or $L(G)$ in $O(m)$ time (For definitions of the tree-width and the wall, we refer the reader to the appendix).

Having Theorem 2, we are ready to describe our $O(mn)$ time algorithm for the edge disjoint path problem more precisely. The algorithm below has appeared in [19], but for the completeness, we include the whole algorithm. We set $h = 4w(k) + 4$, where $w(k)$ is the value given by Theorem 1.

Algorithm for the edge-disjoint paths problem

Step 1. We first apply Theorem 2. If (A) in Theorem 2 occurs, we apply a simple reduction as in (A) and recurse on a smaller graph. If (B) occurs, we apply the standard dynamic programming argument [1, 2]. Thus we may assume outcome (C).

Step 2. If (C) happens, it is possible to throw away a vertex v (*irrelevant vertex*) in the deep inside the wall W if h is large enough (i.e, the vertex in the middle brick of W).

We then recursively apply our algorithm to $G - v$. Since Theorem 2 can be done in $O(m)$ time, the whole algorithm runs in $O(mn)$ time (this improves the time complexity of [31] that gives an $O(m^3)$ algorithm for the edge-disjoint paths problem).

Correctness of the Algorithm

For the correctness of the algorithm, it suffices to prove that v is an irrelevant vertex in Step 2. We now give a proof, which is very similar to that given in [35], Theorem (3.1). We shall essentially reduce the correctness of the algorithm to the unique linkage theorem.

It is easy to see that if G does not have desired k edge-disjoint paths, then $G - v$ does not have them either. Thus it remains to show that if G has desired k edge-disjoint paths in G, then $G - v$ has them as well. Let G' be the line graph $L(G)$. We begin with the following:

(1) The line graph of $X \cup C$ described in (C) of Theorem 2 is still a plane graph.

This is because each vertex in $X \cup C$ has degree at most three in $X \cup C$. Hereafter, let H be the plane subgraph of G' induced by $X \cup C$.

Let $C_1, \dots, C_s$ be disjoint cycles in the plane graph H. Let D_i be the disc in the plane with boundary C_i. We say that they are *concentric* if we have the property that $D_s \subseteq \cdots \subseteq D_1$. Let $C_1, \dots, C_{h/2}$ be concentric cycles in H and $\mathcal{P} = \{P_1, \dots, P_k\}$ be a linkage in G'. Note that since $X \cup C$ contains a wall of height h, it follows that H also contains these $h/2$ concentric cycles.

We assume that the vertices of G' that correspond to the edges incident with v in Step 2 are contained in $D_{h/2} - C_{h/2}$ (again such a choice is possible by the above remark). Let $M = C_1 \cup \cdots \cup C_h$. We only need to prove is the following:

(2) Suppose M exists in G'. Then the desired k vertex-disjoint paths in G' exist such that the vertices of G' that correspond to the edges incident with v in G are not in any of the paths.

This will clearly suffice to complete the proof of the theorem. We prove (2) by induction on the number of vertices of G'. Note that we do not preserve line graph in the inductive step, i.e, when we make a smaller graph and apply induction, it may not be the line graph of some graph. We only require that our graph is contained in H as a subgraph, i.e, $h/2$ concentric cycles in a subgraph of the plane graph H.

Proceeding, if there is a vertex u that is not in $M \cup \mathcal{P}$, then we can delete u from G', and apply induction to $G' - u$. Similarly, consider the case when there exists an edge e that is in C_i, $i \leq t/2$, but one of the endpoints is not used in $\mathcal{P}$. We can contract e and still preserves the existence of concentric cycles $C_1, \dots, C_i/e, \dots, C_{h/2}$ (and a plane subgraph H/e), unless $|C_i| = 3$. But if $|C_i| = 3$, then we can clearly reroute the paths in $\mathcal{P}$ so that they do not touch any vertex inside the disk D_i, except for the vertices in C_i, and so find a linkage avoiding the edges incident v. Thus, after contracting e, we can apply induction to the resulting graph. We conclude that $V(M \cup \mathcal{P}) = V(G')$.

Let $w(k)$ be the value given by Theorem 3. By a *dive* we mean a subpath of a path in $\mathcal{P}$ contained in the disc D_1 with both ends in C_1 and at least one vertex in C_l for some $l \geq w(k) + 1$ We now claim the following:

(3) There are at most $w(k)$ dives.

If there is another linkage $\mathcal{P}'$ equivalent to $\mathcal{P}$ such that $|V(\mathcal{P}')| < |V(\mathcal{P})|$, then there is a vertex u of G' that is not in $\mathcal{P}'$. If u is not in M, then we delete u from G', and apply induction to $G' - u$. Similarly, if u is in M, then there is an edge e with one endpoint u in M. In this case, we contract e as above. After contracting e, we can apply the inductive hypothesis to the resulting graph. Thus we may assume that $\mathcal{P}$ is unique linkage.

We now use the unique linkage theorem to prove (3). Suppose for a contradiction that the linkage $\mathcal{P}$ contains at least $w(k) + 1$ dives. Then since H is a plane subgraph of G' and M is contained in H, there are dives $P_1, P_2, \dots, P_{w(k)+1}$ that are pairwise disjoint and all intersect C_i for $i = 1, \dots, w(k) + 1$. This implies that $P_1, P_2, \dots, P_{w(k)+1}$ all intersect each of $C_1, C_2, \dots, C_{w(k+1)}$, and hence $C_1 \cup P_1, C_2 \cup P_2, \dots, C_{w(k)+1} \cup P_{w(k)+1}$ is a "bramble" in G' of "order" at least $w(k) + 1$ (for the definition of the bramble, we refer the reader to [25]). By [25] the graph G' has tree-width at least $w(k)+1$, a contradiction to the unique linkage theorem. This proves (3).

We are now ready to finish the proof. We claim that no dive intersects $C_{2w(k)+1}$. The *depth* of a dive P is the maximum index i such that $P \cap C_i \neq \emptyset$. To see this, observe that if P is a dive of depth i, then if C_{i-1} does not intersect any path of $\mathcal{P}$, we can reroute the component of $\mathcal{P}$ containing $\mathcal{P}$ to avoid the vertex $P \cap C_i$. Thus, some component (other than the one containing P) of $\mathcal{P}$ intersects C_{i-1}. By planarity, it follows that there exists a dive of depth $i - 1$. Thus, if there exists a dive of depth $2w(k) + 2$, we see that there exist $w(k) + 1$ dives, a contradiction to (3).

If we assume that $h \geq 4w(k) + 4$, we see that no component of $\mathcal{P}$ can intersect $C_{t/2}$. This completes the proof of (2), and the theorem. $\square$

In the next section, we give an outline of our proof of the unique linkage theorem. To help the reader see how the proof goes, we shall give a short proof of the case $k = 2$.

2 Outline of the proof of the Unique Linkage Theorem

The proof proceeds by analyzing what we will call *traversing linkages*. Before we give the exact definition, we first give some intuition of what a traversing linkage is. Let $\mathcal{P}$ be a k-linkage. A linkage $\mathcal{Q}$ traverses $\mathcal{P}$ if when we follow the linkage $\mathcal{Q}$ from beginning to end, we intersect the linkage $\mathcal{P}$ repeatedly in a regular, uniform way. Moreover, these intersections are independent of each other in a sense. That is, in that the first intersections of $\mathcal{P}$ and $\mathcal{Q}$ are contained in a small subpath of $\mathcal{P}$, and $\mathcal{Q}$ never returns to that subpath. We reduce the proof of the Unique Linkage Theorem to showing the following theorem.

Theorem 3. *There exists functions $l(k)$ and $w(k)$ such that the following holds. Let $\mathcal{P}$ be a linkage of order k, and let $\mathcal{Q}$ be a linkage traversing $\mathcal{P}$ of order $w(k)$ and length $l(k)$. Then $\mathcal{P}$ is not unique in $\mathcal{P} \cup \mathcal{Q}$.*

Traversing linkages have two nice properties we use repeatedly in the proof of Theorem 3. First, the graph consists of just two linkages, and so is dramatically simpler than the general graphs typically analyzed in the theory of graph minors. Second, there is an element of symmetry allowing us to move back and forth between analyzing first the linkage $\mathcal{P}$, and then the linkage $\mathcal{Q}$, and back again.

We now give the exact definition of a traversing linkage. We recall that a *ladder of length t* is a graph consisting of two paths of length P_1, P_2 with the vertices of P_i equal to $v_1^i, \dots, v_t^i$ for $i = 1, 2$ as well as edges of the form $v_j 1 v_j 2$ for $1 \leq j \leq t$.

Definition 4. Let $\mathcal{P}$ be a linkage. The linkage $\mathcal{Q}$ *traverses* $\mathcal{P}$ (or, equivalently, is a traversing linkage) if there exist disjoint subpaths $B_1, \dots, B_l$ in $\mathcal{P}$ such that the following hold:

a. The linkage $\mathcal{Q}$ intersects $\mathcal{P}$ only in the subpaths $B_1, \dots, B_l$, i.e. $V(\mathcal{Q}) \cap V(\mathcal{P}) \subseteq \bigcup_1^l V(B_i)$.

b. For all $Q \in \mathcal{Q}$ and $1 \leq i \leq l$, $Q \cap B_i$ is a (possibly trivial) subpath of B_i.

c. For every element $Q \in \mathcal{Q}$, we may traverse the path Q from one end to the other, we encounter the paths $B_1, B_2, \dots, B_l$ in that order.

d. If we look at the Z the set of subpaths of $\mathcal{Q}$ with one end in B_i and another end in B_{i+1} for $1 \leq i \leq l - 1$, then $Z \cup B_i \cup B_{i+1}$ forms a subdivided ladder after possibly deleting vertices of degree one.

The paths $B_1, B_2, \dots, B_l$ are called the *basis subpaths* of the traversing linkage $\mathcal{Q}$. The value l is the *length* of the traversing linkage $\mathcal{Q}$. Again, the order of the traversing linkage is the number of components. Fix labels s_P, t_P to the endpoints of every component $P \in \mathcal{P}$. If we consider the ladder in d, there are two distinct possibilities. Let P and P' be the components of $\mathcal{P}$ containing B_i and B_{i+1}, respectively. We say that $\mathcal{Q}$ *twists between B_i and B_{i+1}* if for all j, the j^{th} component of Z we intersect traveling P from s_P to t_P is the $(w - j + 1)^{th}$ when traversing P' from $s_{P'}$ to $t_{P'}$ (where w is the order of $\mathcal{Q}$).

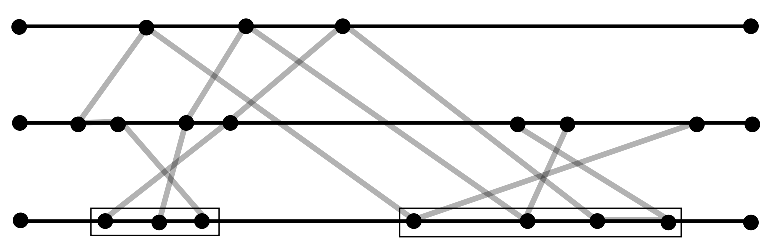

Figure 1: The grey linkage is a traversing linkage of order three and length five traversing the black linkage. The first and fourth basis subpaths are indicated with boxes. The grey linkage twists between the first and second basis subpaths.

Observe, that if we swap the labels s_P and t_P on a component $P \in \mathcal{P}$, then if we consider some basis subpath B_i contained in P, if $\mathcal{Q}$ twists between B_i and B_{i+1} before the swap, then $\mathcal{Q}$ will not twist between the two basis subpaths after the swap (and vice versa: if $\mathcal{Q}$ does not twist before the swap, then it will twist after the swap).

We briefly describe now how we reduce the Unique Linkage Theorem to the proof of Theorem 3. The analysis is somewhat similar to the proof of the edge disjoint version of the disjoint paths problem. We proceed in two basic steps. First, we pick a prospective counter-example to the Unique Linkage Theorem: a linkage $\mathcal{P}$ contained in a graph G with $V(G) = V(\mathcal{P})$ such that G does not contain an equivalent linkage on fewer vertices. Moreover, we make the assumption that the tree width of the graph is huge. We first show that such a counterexample G cannot contain a large clique minor, and then using what we call the Weak Structure Theorem, we show that there exists a large planar subgraph H containing a huge wall such that the linkage $\mathcal{P}$ interacts with H planarly. In other words, even if there exist vertices with neighbors in the center of H, the components of $\mathcal{P}$ intersect H in a way that always respects the planar embedding of H. We then, as in the proof of the edge disjoint version, take a large number of concentric cycles such that they intersect the linkage $\mathcal{P}$ in a clean way. The concentric cycles as they travel through the linkage $\mathcal{P}$ will then provide the traversing linkage $\mathcal{Q}$.

The proof of Theorem 3 will be the main work in our proof of the Unique Linkage Theorem. The remainder of this section will be devoted to a brief outline of the proof of Theorem 3.

The proof of Theorem 3 proceeds by finding many sublinkages in $\mathcal{Q}$ forming what we will call $\mathcal{Q}$-bumps. Let $\mathcal{P}$ be a linkage and $\mathcal{Q}$ be a traversing linkage of order w. Let $B_1, \dots, B_l$ be the basis subpaths. A *$\mathcal{Q}$-bump* is a sublinkage $\overline{\mathcal{Q}}$ of $\mathcal{Q}$ of order w such that there exist indices i and i' and a path $P \in \mathcal{P}$ such that

a. every component of $\overline{\mathcal{Q}}$ has one endpoint in B_i and one endpoint in $B_{i'}$ and no internal vertex in P, and

b. B_i and $B_{i'}$ are both contained in P.

$\mathcal{Q}$-bumps can be thought of as a cylindrical set of subpaths wrapping around a sublinkage of the linkage $\mathcal{P}$. A $\mathcal{Q}$ bump allows one to reroute the linkage $\mathcal{P}$ - not to find an equivalent linkage - but rather to cyclically shift by one some subset of the paths. We make this more explicit in the following observation.

Observation 5. *Let $\mathcal{P}$ be a linkage of order k with components P_i for $1 \leq i \leq k$. Let s_i and t_i be the endpoints of P_i for $1 \leq i \leq k$. Let $\mathcal{Q}$ be a traversing linkage of order $k+1$ with basis subpaths $B_1, \dots, B_l$. Let $\overline{\mathcal{Q}}$ be a $\mathcal{Q}$-bump of length $l'+1$ with basis subpaths $\overline{B}_1, \dots, \overline{B}_{l'+1}$ satisfying the following properties.*

i. Assume $\overline{B}_i$ is contained in P_i for $1 \leq i \leq l'$. Specifically, note $\overline{B}_i$ and $\overline{B}_{i'}$ are contained in distinct components of $\mathcal{P}$ for $1 \leq i < i' \leq l'$.

ii. For all i, $1 \leq i < l'+1$, $\mathcal{Q}$ does not twist between B_i and B_{i+1}.

Then $\mathcal{P} \cup \overline{\mathcal{Q}}$ contains disjoint paths $P'_1, \dots, P'_k$ such that the endpoints of P'_i are s_i and t_{i+1} for $1 \leq i \leq l'$ (taken modulo l') and the ends of P_i are s_i and t_i for $i > l'$. Moreover, the paths $P'_1, \dots, P'_k$ can be chosen to avoid some vertex of $\mathcal{P}$.

We illustrate the observation in Figure 2.

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two vertex disjoint $\mathcal{Q}$-bumps, and assume that there exists a component P of $\mathcal{P}$ containing all the endpoints of $\mathcal{R}_1$ and $\mathcal{R}_2$. Let $\pi(i)$ and $\sigma(i)$ be values, for $i = 1, 2$ such that $\mathcal{R}_i$ has endpoints in $B_{\pi(i)}$ and $B_{\sigma(i)}$. Consider the subpath of P_i of P connecting the ends of R_i, i.e. let P_i be the minimal subpath of P containing $B_{\pi(i)}$ and $B_{\sigma(i)}$ for $i = 1, 2$. Then $\mathcal{R}_1$ and $\mathcal{R}_2$ are *independent* if P_1 and P_2 are disjoint.

We now have everything in place to outline the argument for the proof of Theorem 3. Let $\mathcal{P}$ be our linkage of order k which we would like to reroute, and let $\mathcal{Q}$ be a *very* long traversing linkage of *huge* order. Fix a path P_1 in the linkage $\mathcal{P}$.

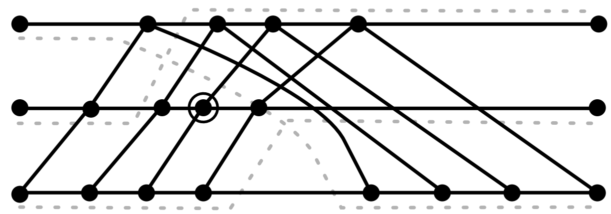

Figure 2: A bump of order 4 traversing a linkage of order 3. The rerouted paths guaranteed by Observation 5 are indicated as dotted paths. Note that the circled vertex is avoided by the new linkage.

First, we observe that for some component P_1 of $\mathcal{P}$, the linkage $\mathcal{Q}$ must return frequently to the path P.

If there existed many disjoint independent $\mathcal{Q}$-bumps with ends in P_1, it would be relatively easy to ensure that some subset large satisfies the necessary properties to apply Observation 5. Then using at most k bumps satisfying the conditions of Observation 5, we repeatedly rotate and find an equivalent linkage avoiding some vertex.

The difficulty is how to find many disjoint independent $\mathcal{Q}$-bumps. We switch perspectives at this point and consider how the components of $\mathcal{P}$ intersect with $\mathcal{Q}$. The path P_1 repeatedly crosses the linkage $\mathcal{Q}$, always hitting the paths of $\mathcal{Q}$ in order. Then there are two possible cases. First, it is possible that some subpath P_1' of P_1 intersects $\mathcal{Q}$ in a somewhat regular manner, yielding many disjoint, independent $\mathcal{Q}$-bumps on P_1'. In this case, we apply Observation 5 to find a linkage equivalent to $\mathcal{P}$ avoiding some vertex. Alternatively, no such path P_1' exists and the path P_1 intersects $\mathcal{Q}$ in a more complex fashion. In this case we are able to find a large complete minor in $\mathcal{P} \cup \mathcal{Q}$. As we have already discussed, a large clique minor allows us to find an equivalent linkage avoiding some vertex of $\mathcal{P}$. In each case, we find an equivalent linkage avoiding some vertex of $\mathcal{P}$, completing the proof.

We expand, for a moment, on these ideas for interested readers with some familiarity with the graph minors techniques. We return to a more general discussion below. Let $\mathcal{P}$ be a linkage of order k and $\mathcal{Q}$ be a traversing linkage of order w and length l. Let the basis subpaths of $\mathcal{Q}$ be $B_1, B_2, \dots, B_l$.

We contract all the edges incident a vertex of degree at most two, as well as edges of $E(\mathcal{Q}) \cap E(\mathcal{P})$. Thus we may assume that:

1. there are no edges contained in $\mathcal{Q} \cap \mathcal{P}$, and
2. there are no vertices in $V(\mathcal{Q}) \setminus V(\mathcal{P})$.

We can fix a labeling $Q_1, Q_2, \dots, Q_w$ of the components of $\mathcal{Q}$ of order w, so that every component $P \in \mathcal{P}$ satisfies the following. The path P can be decomposed into subpaths $R_1, \dots, R_t$ and edges $e_1, \dots, e_{t-1}$ that are pairwise disjoint so that e_i connects the ends of R_i and R_{i+1} and each R_i has one end in Q_1, the other end in Q_w, and intersects the paths of $\mathcal{Q}$ in order, i.e. $Q_1, Q_2, \dots, Q_w$. (In fact, the paths R_i are the basis subpaths of $\mathcal{Q}$ on P, ordered by traversing P).

If we look at the graph formed by $P \cup \mathcal{Q}$, we see that $\mathcal{Q} \cup \bigcup_1^t R_i$ forms a subdivision W of the $(w \times t)$-grid. The horizontal paths of the grid are the Q_i for $1 \le i \le w$, and the vertical paths are the R_i, $1 \le i \le t$. The edges e_i form a matching with the ends contained in $Q_1 \cup Q_w$. We keep the grid W aside, and for the rest of the proof focus on the edges e_i. As we described above, there are essentially two cases. If these edges are relatively well behaved, all but a small number of them can be embedded in a low genus surface. In this case, this will allow us to reroute the linkage $\mathcal{P}$ avoiding some vertex using techniques of Robertson and Seymour for the disjoint paths problem in the bounded genus case [29]. Alternatively, the edges e_i are not tame. In this case, we will find a large clique minor. Then, again we see that $\mathcal{P}$ can again be rerouted to avoid some vertex v.

The edges e_i, together with the outer face boundary of W, comprise a *society*, one of the key topics in Graph Minors Theory [30]. Our proof builds on results in [30], and extending them in such a way that the outcomes include "genus addition", i.e, a handle addition and a crosscap addition. We further adapt some ideas in [32, 33] to grow a graph on the surface with large representativity. This process stops when we have a huge clique minor, because, as mentioned above, if there is a huge clique minor, we can reroute the linkage $\mathcal{P}$ to avoid some vertex. We reiterate that our proof does not need most of the heavy machinery in Graph Minor theory. This is for several reasons. First, because our society consists of only a matching, the analysis is simplified. Second, we do not have to worry about global connectivity issues as the society vertices are the vertices of the outer ring of a grid. And, finally, certain degenerate cases will allow us to easily find many disjoint $\mathcal{Q}$-bumps, an outcome not available in the general graph minors arguments. This final point will allow us to evade the topic of "vortices", a major savings in time and effort. Further ingredients of the graph minors series which we are able to avoid include, "embedding up to 3-separations", "tangle, respectful tangle" etc.

We return now to a more general overview of the proof. A technicality we ignored in this outline is the following. Given that we find many disjoint independent $\mathcal{Q}$-bumps on the subpath P_1' of P_1, how do use the bumps to reroute the linkage? We do so by *splitting* on edges. Given an edge e in a linkage $\mathcal{P}$, we say that the linkage $\mathcal{P} - e$ is obtained from $\mathcal{P}$ by splitting $\mathcal{P}$ on e. We note that the property of being a unique linkage is preserved upon splitting a linkage on a given edge:

Observation 6. *Let $\mathcal{P}$ be a unique linkage in a graph G, and let $\overline{\mathcal{P}}$ be obtained from splitting $\mathcal{P}$ on some edge e. Then $\overline{\mathcal{P}}$ is a unique linkage in G. Moreover, if $\mathcal{Q}$ is a traversing linkage of $\mathcal{P}$ of length l, with basis subpaths $B_1, B_2, \dots, B_l$, then if $e \notin E(B_i)$ for all $1 \leq i \leq l$, then $\mathcal{Q}$ is a traversing sublinkage of $\overline{\mathcal{P}}$.*

Thus we perform possibly two edge splits on $\mathcal{P}$ to obtain a new linkage with a component equal to P_1'. Given that we have many disjoint, independent $\mathcal{Q}$ bumps attaching to P_1', we contradict that the new linkage is unique, and by the observation, that the original linkage $\mathcal{P}$ is unique.

In conclusion, we give a complete proof of the $k = 2$ case of Theorem 3. Robertson and Seymour [34] observe that this can be easily shown directly; however, we give a proof using traversing linkages in the hopes that it further illustrates the tools and techniques of the main result.

Theorem 7. *Let $\mathcal{P}$ be a linkage of order 2. Let $\mathcal{Q}$ be a traversing linkage of order five and length 33. Then there exists a vertex $v \in V(\mathcal{P})$ such that $(\mathcal{P} \cup \mathcal{Q}) - v$ contains a linkage $\mathcal{P}'$ equivalent to $\mathcal{P}$.*

Proof. Let the components of $\mathcal{P}$ be P_1 and P_2, and label the ends of P_i s_i and t_i for $i = 1, 2$. Let the basis subpaths of $\mathcal{Q}$ be $B_1, B_2, \dots, B_{33}$. We assume, to reach a contradiction, that there do not exist paths P_1', P_2' that avoid some vertex of $\mathcal{P}$ such that the endpoints of P_i' are s_i and t_i.

The linkage $\mathcal{Q}$ repeatedly passes back and forth between P_1 and P_2. To simplify the picture somewhat, we consider the following auxiliary graph H with vertices equal to the set of basis subpaths B_i for $1 \leq i \leq 33$ and two vertices B_i and B_j connected by an edge if either there is a subpath of P_i connecting them avoiding all other basis subpaths, or if $|j - i| = 1$. It follows that $E(H)$ is comprised of three edge disjoint paths: one for each of the paths P_i and one for the linkage $\mathcal{Q}$. The edges of H of the form B_iB_{i+1} have two distinct types - the linkage $\mathcal{Q}$ can either twist between B_i and B_{i+1} or not. We will refer to the edges B_iB_{i+1} where $\mathcal{Q}$ twists as the *odd* edges of H; every other edge of H will be called *even.* If Σ is the set of odd edges, then by swapping the labels s_i and t_i, the resulting set of odd edges is $\Sigma \triangle X$ where $\triangle$ denotes the symmetric difference and X is the set of all edges of the form B_iB_{i+1}.

First, observe that there does not exist an index i such both the edge $B_{i-1}B_i$ and B_iB_{i+1} are even in H. Otherwise, we can find paths P_1' and P_2' equivalent to $\mathcal{P}$ avoiding an internal vertex of B_i. Similarly, there does not exist an index i such that $\mathcal{Q}$ does twist between both B_{i-1} and B_i and between B_i and B_{i+1}. This is because we could swap the labels on the ends P_1 so that $\mathcal{Q}$ does not twist between B_{i-1} and B_i and between B_i and B_{i+1}. If we let R be the path in H of consisting of the edges of the form B_iB_{i+1} for $1 \leq i \leq 32$, it follows that the edges of R alternate between edges in Σ and edges not in Σ.

Also, observe that for indices i and j such that both the edges B_iB_{i+1} and B_jB_{j+1} are not in Σ, we have that the edges do not "cross" in H. That is, if B_i and B_j are both contained P_1, say, and occur on P_1 in that order when traversing from s_1 to t_1, then B_{i+1} occurs before B_{j+1} on P_2 when traversing from s_2 to t_2. Otherwise, we would be able to find an equivalent linkage avoiding some vertex of $\mathcal{P}$. Similarly, if i and j are two indices such that both the edges B_iB_{i+1} and B_jB_{j+1} are in Σ, and if both B_i and B_j are contained in P_1 in that order, then it follows that B_{j+1} and B_{i+1} occur on that order when traversing P_2 from s_2 to t_2.

After possibly swapping the labels s_i and t_i for one or possibly both values of i, we see by examining the graph H that the following must hold for $\mathcal{Q} \cup \mathcal{P}$. Traversing P_1 from s_1 to t_1, we see $B_2, B_6, \ldots, B_{26}, B_{30}, B_{32}, B_{28}, \ldots, B_8, B_4$ in that order, and traversing P_2 from s_2 to t_2, we see $B_1, B_5, \ldots, B_{29}, B_{33}, B_{31}, B_{27}, \ldots, B_7, B_3$ in that order. Moreover, $\mathcal{Q}$ twists between B_i and B_{i+1} if and only if i is even. Let e_1 be an edge of P_1 separating B_{30} from B_{32}, and let e_2 be an edge of P_2 separating B_{33} from B_{31}. If we split the linkage $\mathcal{P}$ on e_1 and e_2, we have a linkage $\mathcal{P}'$ of order four. Moreover, by appropriately choosing the labels for the endpoints of components of $\mathcal{P}'$, we may assume that $\mathcal{Q}$ does not twist between any two basis subpaths in $\mathcal{P}'$. We label the components of $\mathcal{P}'$ as P'_i for $1 \leq i \leq 4$ such that the endpoints of $\mathcal{Q}$ are contained in P'_1.

By examination, we see that there exist four disjoint independent $\mathcal{Q}$-bumps with endpoints in P'_1 satisfying *i.* and *ii.* in Observation 5. Moreover, if we let the endpoints of P'_i be s'_i and t'_i, then for each such bump, the rerouting guaranteed by Observation 5 results in paths with endpoints s'_i and t'_{i+1}. By using all four re-routings, we see that we wrap around and find a linkage $\overline{\mathcal{P}}$ equivalent to $\mathcal{P}'$ avoiding some vertex of $\mathcal{P}'$. Then Observation 6 implies a contradiction to our choice of $\mathcal{P}$ to be a unique linkage, proving the claim. $\square$

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A Outline of the appendix

In this appendix, we present the proofs of Theorems 1 and 3. We now give a brief overview of the sections. In the next section, we present the Graph Minors tools which we will need for the argument. In Section C, we prove a necessary technical lemma about combining a large family of traversing linkages into a single traversing linkage. This will be necessary in order to generalize Observation 5 and weaken the conditions *i* and *ii*. In Section D, we show how a suitable prospective counterexample to the Unique Linkage Theorem guarantees the the existence of a large traversing linkage. In this section, we include the proof of the Unique Linkage Theorem assuming Theorem 3. The remainder of the appendix is concerned with the proof of Theorem 3.

Section E covers much of the same ideas laid out in the outline of the proof in the main article. Specifically, it shows how we reduce the proof of Theorem 3 to one of analyzing societies consisting of a matching glued onto the outside ring of a grid. Sections F, G, and H deal with several independent results on societies consisting of only a matching, culminating in the main lemma we will use to analyze these societies in the proof of Theorem 3. This lemma is presented in Section H. Section I provides a final lemma necessary to the proof of Theorem 3, which is presented in the last section.

B Graphs in surfaces and the Weak Structure Theorem

Before developing the algorithm for the k disjoint paths problem in general graphs, Robertson and Seymour first studied the problem in graphs embedded in some fixed surface. We will use these tools as well, and we present them in this section. We will also use a weak structure theorem for graphs without a large clique minor. We present this theorem at the end of this section.

We begin with several basics on graphs embedded in surfaces. By surface we mean a 2-manifold without boundary, and we always assume the embedded graph has a 2-cell embedding. If a graph G is embedded in a surface Σ not equal to the sphere, the *representativity* of the embedding is the minimal number of points in which a homotopically non-trivial curve C in the surface intersects the embedded graph.

Again, let G be a graph embedded in a surface Σ and let F be a face of G bounded by a cycle C. Let the vertices of C be labeled $v_1, v_2, \ldots, v_k$ such that they occur in that cyclic order on C. A *rooted circular grid on the face F of depth t* consists of pairwise disjoint contractable cycles $C_1, \ldots, C_t$ in G and pairwise disjoint paths $P_1, \ldots, P_k$ such that:

- each C_i defines a disc D_i in Σ with $D_t \supseteq D_{t-1} \supseteq \cdots \supseteq D_1 \supseteq F$;
- P_i has one end equal to v_i and the other end in C_t;
- $P_i \cap C_j$ is a subpath of P_i for all $1 \leq i \leq k$, $1 \leq j \leq t$.

Rooted circular grids will be essential later in the proofs, as they allow us to maintain representativity when adding edges to a fixed face. Before explaining this further, we need several background definitions.

A *society* is a pair (G, Ω) where G is a graph and Ω is a cyclic ordering of a subset of the vertices of G. The set of vertices ordered by Ω are referred to as the *society vertices*. We will often use Ω to refer both to the cyclic ordering as well as the set of society vertices.

We recall that for a graph G and a set A of vertices in G, an *A-path* in G is a path with both endpoints in A and no internal vertex in A.

A *k-crosscap* in a society (G, Ω) consists of k disjoint Ω-paths $P_1, \ldots, P_k$ such that the ends of P_i labeled s_i and t_i such that the vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ occur in Ω in that order. A *k-handle* consists of $2k$ disjoint Ω-paths $P_1, \ldots, P_k$, $Q_1, \ldots, Q_k$ such that the ends of P_i are s_i and t_i for $1 \ldots, t$ and the ends of Q_i are u_i and v_i such that the vertices $s_1, \ldots, s_k, u_1, \ldots, u_k, t_k, \ldots, t_1, v_k, \ldots, v_1$ occur in Ω in that order. See Figure 3.

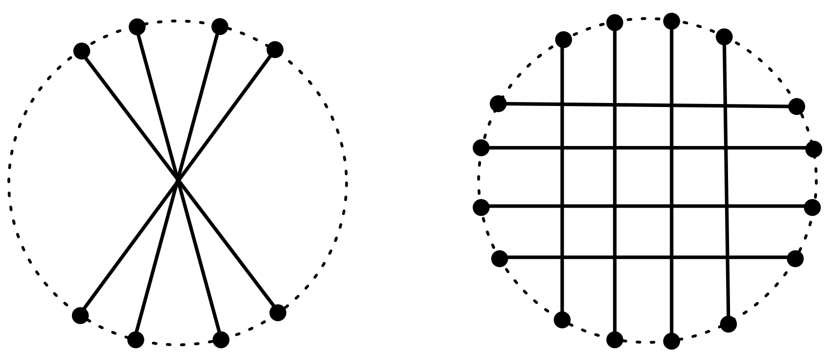

Figure 3: An example of a 4-crosscap and a 4-handle.

Observation 8. *Let G be a graph embedded in a surface Σ of representativity t. Let F be a face with boundary cycle C, and assume there exists a rooted circular grid of depth t rooted at F. Let Ω be the natural cyclic ordering of $V(C)$ given by following the cycle C in clockwise order in the face F. Let H be a set of edges (H, Ω) be a set of edges forming a t-crosscap (or a t-handle). Then $G \cup H$ embeds with representativity t in the surface Σ' equal to Σ plus a crosscap in F (or Σ' is equal to a Σ plus a handle* *in F). Moreover, if F' is any face of the embedding in Σ bounded by a subset of the vertices of Ω, then there exists a rooted circular grid of depth $\lfloor t/2 \rfloor$ rooted at F'.*

The proof of Observation 8 follows from the fact that any homotopically non-trivial curve in Σ' (that is homotopically trivial in Σ) must in fact be contained the disc bounded by the circular grid. The statement then follows from the case when we add a t-handle or t-crosscap to the inner face of a circular grid.

We will need the following two results or Robertson and Seymour.

Theorem 9 ([29], Theorem (9.1))**.** *For all $t \geq 1$ and for all surfaces Σ in which K_t can be embedded, there exists a value $f(t, \Sigma)$ such that if G is a graph embedded in Σ with representativity $f(t, \Sigma)$, then G contains K_t as a minor.*

The next result is essentially Theorem 1 for graphs embedded in a surface of bounded genus.

Theorem 10 ([29], follows from Theorem (7.5))**.** *There exists a function $w = w(k, \Sigma)$ such that the following holds. Let $\mathcal{P}$ be a linkage in a graph G with $V(G) = V(\mathcal{P})$. Assume G embeds in Σ. If the tree width of G is at least w, then there exists a vertex $v \in V(G)$ and a linkage $\mathcal{P}'$ equivalent to $\mathcal{P}$ in $G - v$.*

We believe that we also have short proofs for both Theorems 9 and 10. This allows us to avoid using [29], which is also lengthly.

We finish this section with the the version of the structure theorem we will use. Before describing the theorem, we first need the definition of a *wall*, as well as some notation.

For positive even integers r, define a graph H_r as follows. Let $P_0, \ldots, P_r$ be r vertex disjoint ('horizontal') paths of length $2r+1$, say $P_i = v_0^i \ldots v_{2r+1}^i$. Let $V(H_r) = \bigcup_{i=1}^r V(P_i) \setminus \{v_0^0, v_{2r+1}^r\}$, and let

$$E(H_r) = \left(\bigcup_{i=1}^{r} E(P_i) \setminus \{v_0^0 v_1^0, v_{2r}^r v_{2r+1}^r\} \right) \cup \left\{ v_j^i v_j^{i+1} : \ i \text{ odd}, \ j \text{ even}; \ 1 \leq i < r; \ 0 \leq j \leq 2r+1 \right\}$$
$$\cup \left\{ v_j^i v_j^{i+1} : \ i \text{ even}, \ j \text{ odd}; \ 0 \leq i < r; \ 1 \leq j \leq 2r+1 \right\}.$$

The 6-cycles in H_r are its *bricks*. In the natural plane embedding of H_r, these bound its 'finite' faces. The outer cycle of the unique maximal 2-connected subgraph of H_r is the *boundary cycle* of H_r. Any subdivision $H = TH_r$ of H_r will be called an *r–wall* or a *wall of size* r. The *bricks* and the *boundary cycle* of H are its subgraphs that form subdivisions of the bricks and the boundary cycle of H_r, respectively.

Let us recall that an *r-grid* is a graph which is isomorphic to a subdivision of the graph W_r obtained from the Cartesian product of paths $P_r \times P_r$, with vertex set $V(W_r) = \{(i,j) \mid 1 \leq i \leq r, \ 1 \leq j \leq r\}$ in which two vertices (i,j) and (i',j') are adjacent if and only if one of the following possibilities holds:

(1) $i' = i$ and $j' \in \{j-1, j+1\}$.

(2) $j' = j$ and $i' = i + (-1)^{i+j}$.

Let us recall that the $(a \times b)$-grid can be defined in a similar way.

A *tree-decomposition* of a graph G is a pair (T, W), where T is a tree and W is a family $\{W_t \mid t \in V(T)\}$ of vertex sets $W_t \subseteq V(G)$, such that the following two properties hold:

(W1) $\bigcup_{t \in V(T)} W_t = V(G)$, and every edge of G has both ends in some W_t.

(W2) If $t, t', t'' \in V(T)$ and t' lies on the path in T between t and t'', then $W_t \cap W_{t''} \subseteq W_{t'}$.

The *tree-width* of G is defined as the minimum width taken over all tree decompositions of G.

One of the most important results concerning the tree-width is that it guarantees the existence of a large wall. We give an algorithmic version of this result, which is due to Bodlaender [2].

Theorem 11. *For any constant r, there exists a constant $w = f_1(r)$ satisfying the following: There exists an $O(w^w n)$ time algorithm that, given a graph G, either finds a tree-decomposition of G of width w or finds a wall W of height r. For a planar graph, the time complexity can be improved to $O(2^w n)$.*

Definition 12. For a positive integer r and a graph G, a *flat r-wall decomposition of G* is a collection of subgraphs $G_0, G_1, \dots, G_n$ and an r-wall subgraph W with boundary cycle C satisfying the following:

a. $G = \bigcup_0^n G_i$,

b. $V(G_i) \cap V(G_j) \subseteq V(G_0)$ for all $1 \le i < j \le n$ and $|V(G_i) \cap V(G_0)| \le 3$ for all $1 \le i \le n$, and

c. for all $1 \le i \le n$, $V(G_i) \setminus V(G_0)$ contains at most one vertex of degree 3 in W.

Let G_0' be the subgraph resulting from G_0 after adding an edge to any two nonadjacent vertices u and v contained in $V(G_0) \cap V(G_i)$ for some index $1 \le i \le n$.

d. The graph G_0' is planar and can be embedded such that the infinite face is bounded by $V(C) \cap V(G_0)$.

Recall that $\partial(X)$ for any subset X in a graph G is the set of vertices v in $V(G) \setminus X$ such that v has a neighbor in X.

We are give the Weak Structure Theorem.

Theorem 13 (Weak Structure Theorem, [31], Theorem (9.4))**.** *For all $k \ge 1$, r even, there exists a value $w = w(t, r)$ and α such that the following holds. Let G be a graph that does not contain K_t as a minor with tree-width at least w. Then there exists a set $A \subseteq V(G)$ with $|A| \le \alpha$ such that $G - A$ can be decomposed into subgraphs H and G' with $G - A = G' \cup H$. Moreover, the graph H has a flat r-wall decomposition $H_0, H_1, \dots, H_n$, W and C satisfying:*

i) $\partial_{G-A}(V(G')) \subseteq V(C) \cap V(H_0)$, *and*

ii) *every vertex of degree 3 in W is contained in $V(H_0)$.*

The weak structure theorem is in fact weaker than the full structure theorem. Notice that a graph G may have the desired decomposition in the statement of the weaker theorem, and yet still have an arbitrarily large clique minor. Instead, in the full structure theorem of the form of the clique-sum decomposition, for all integers t there exists a value T such that any graph with no K_t minor has a certain structure, and, moreover, any graph which does have this structure cannot contain K_T as a minor. See [7].

C Merging traversing linkages

The goal of this section is a lemma showing that given many distinct linkages $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ traversing a fixed linkage $\mathcal{P}$, either we are able to reroute the linkage $\mathcal{P}$ to avoid a vertex, or we can join sublinkages of many of the $\mathcal{Q}_i$ together to find a new traversing linkage. This lemma will be necessary in the coming sections.

We will need the following two easy lemmas. The first is a restatement of a result of Robertson and Seymour; the second follows from a lemma of Erdős and Szekeres.

Let $P = p_1, \cdots, p_l$ be a path. We say that two edge $e_1 = p_a p_b$ and $e_2 = p_c p_d$ *cross*, if either $a < c < b < d$ or $c < a < d < b$. Otherwise we say that e_1 and e_2 do not cross.

Lemma 14 ([29])**.** *Let P be a path and let $e_1, \ldots, e_n$ be n disjoint edges with endpoints in $V(P)$ (but each e_i is not an edge of P). If $n \geq k^3$, then there exist k distinct e_i such that they pairwise cross, they define pairwise disjoint intervals of P, or they define nested intervals of P.*

Let $P = p_1, \cdots, p_l$ and $Q = q_1 \cdots q_l$ be two disjoint paths. We say that two edge $e_1 = p_a q_b$ and $e_2 = p_c q_d$ *cross*, if either $a < c$ and $d < b$ or $c < a$ and $b < d$. Otherwise we say that e_1 and e_2 do not cross.

Lemma 15. *Let P and Q be two paths, and let $e_1, \ldots, e_n$ be disjoint edges, each with one end in P and one end in Q. If $n \geq k^2$, then there exist k edges e_i which pairwise cross, or pairwise do not cross.*

We now give the main lemma in this section. It says that given many distinct linkages each traversing a fixed linkage $\mathcal{P}$, we can find a subset of them for which a subset of their paths have the same behavior on $\mathcal{P}$.

Lemma 16. *For all positive integers k, l, n, and r, there exists positive integers N and R satisfying the following. Let $\mathcal{P}$ be a linkage of order k and let $\mathcal{Q}_1, \ldots, \mathcal{Q}_N$ be linkages traversing $\mathcal{P}$, each of order R and length l. Then either*

1. *there exists indices $\pi(1), \ldots, \pi(n)$ and linkages $\mathcal{R}_{\pi(i)} \subseteq \mathcal{Q}_{\pi(i)}$ for $1 \leq i \leq n$, each of order r and length l, such that $\bigcup_1^n \mathcal{R}_{\pi(i)}$ is a linkage traversing $\mathcal{P}$, or*
2. *there exists a linkage $\mathcal{P}'$ in $\mathcal{P} \cup \left(\bigcup_1^N \mathcal{Q}_i\right)$ equivalent to $\mathcal{P}$ and avoiding a vertex $v \in V(\mathcal{P})$.*

Proof. Intuitively, we will begin with a subset of the linkages $\mathcal{Q}_i$ all of which start on the same component of $\mathcal{P}$. Then we traverse the $\mathcal{Q}_i$'s from beginning to end. At each step we will find a large subset that has the same behavior for the first t intersections with $\mathcal{P}$, or we will use Observation 5 to reroute the linkage $\mathcal{P}$ to avoid some vertex.

We think of t as indexing the number of steps we have proceeded so far in the argument. We define the following. Let $\mathcal{P}(t)$ be a linkage of order at most $k + t$ obtained by a series of at most t splits on edges of $\mathcal{P}$. Let $I(t) \subseteq \{1, \ldots, N\}$ be a subset of $N(t)$ indices, and let linkages $\mathcal{R}(t)_i$ sublinkages of $\mathcal{Q}_i$ for $i \in I(t)$ where the basis subpaths of $\mathcal{R}(t)_i$ are $B(t)_i^j$ for $1 \leq j \leq l$ satisfying the following:

a. The basis subpaths are pairwise disjoint, i.e. $B(t)_i^j \cap B(t)_{i'}^{j'} = \emptyset$ for all i, i', j, j' unless $i = i'$ and $j = j'$.

b. There exist distinct components $S_1, \ldots, S_t$ of $\mathcal{P}(t)$ such that $B(t)_i^j \subseteq S_j$ for all $1 \leq i \leq N(t)$, $1 \leq j \leq t$.

c. Each of the $\mathcal{R}(t)_i$ has order $k + l + 1 + r$.

d. The $\mathcal{R}(t)_i$ are either pairwise crossing or pairwise non-crossing between S_j and S_{j+1} for $1 \leq j \leq t - 1$. In other words, for all $1 \leq j \leq t - 1$, there exists i and i' with $B(t)_i^j$ before $B(t)_{i'}^j$ on S_j and $B(t)_i^{j+1}$ before $B(t)_{i'}^{j+1}$ if and only if this holds for all indices $i, i' \in I(t)$.

If we prove the existence of such linkages with $t = l$ and $N(l) \geq n$, then the linkages $\mathcal{R}(l)_i$ for $i \in I(l)$ will be the desired linkages in the statement of the lemma, with the subpaths $S_1, \ldots, S_l$ forming the basis subpaths of the traversing linkage. We will give a recursive relation for $N(i)$ which will indicate how large N must be in order to make the lemma true.

As a first step, we uncross the basis subpaths of the various linkages $\mathcal{Q}_i$, $1 \leq i \leq N$. Let the basis subpaths of $\mathcal{Q}_i$ be B_i^j for $1 \leq j \leq l$. Note that for indices i, i', j, and j' such that $B_i^j \cap B_{i'}^{j'} \neq \emptyset$, there exist disjoint paths $\overline{B}_i^j \subseteq B_i^j$ and $\overline{B}_{i'}^{j'} \subseteq B_{i'}^{j'}$ such that each of $\overline{B}_i^j$ and $\overline{B}_{i'}^{j'}$ contain half (rounded down) of the components of $\mathcal{Q}_i \cap B_i^j$ and $\mathcal{Q}_{i'} \cap B_{i'}^{j'}$, respectively. If we apply this to every possible choice of i, i', j, and j', we see that if we assume

$$R \geq (k + l + 1 + r)3^{N^2 l^2}$$

then there exist linkages $\overline{\mathcal{Q}}_i \subseteq \mathcal{Q}_i$ for $1 \leq i \leq N$ with basis subpaths $\overline{B}_i^j$ for $1 \leq j \leq l$, each of order $(k + l + 1 + r)$ such that the basis subpaths are pairwise disjoint.

We first construct $\mathcal{P}(1)$, $I(1)$, $\mathcal{R}(1)_i$, $B(1)_i^j$, and S_1 as follows. There exists a component S_1 of $\mathcal{P}$ such that at least N/k of the components of $\overline{\mathcal{Q}}_i$ have $\overline{B}_i$ contained in S_1. We let $\mathcal{P}(1) = \mathcal{P}$, $I(1)$ the set of indices with $\overline{B}_i^1$ contained in S_1 (and let $B(1)_i^j = \overline{B}_i^j$ in general), and, finally, $R(1)_i = \overline{\mathcal{Q}}_i$.

Let $\mathcal{P}(t)$, $I(t)$, $S_1, \ldots, S_t$, $\mathcal{R}(t)_i$ for $i \in I(t)$ with basis subpaths $B(t)_i^j$ for $1 \leq j \leq k + l + 1 + r$ be given in order to calculate the corresponding $t + 1$ structures. We will show that we can find a desired subset of indices with

$$N(t+1) \geq \left(\frac{N(t)}{k+l}\right)^{\frac{1}{3}}.$$

We will additionally need the assumption that $N(t) \geq 2l + 1$. If we look at the path S_t, then there exists some component $P \in \mathcal{P}(t)$ such that P contains at least $N(t)/(k+l)$ of the "next" basis subpaths $B(t)_i^{t+1}$. We let I be the indices i with $B(t)_i^{t+1}$ contained in P.

There are now two cases to consider: if $P = S_i$ for some index i, or alternatively, $P \in \mathcal{P}(t) - \left(\bigcup_1^t S_i\right)$. The easier case is when $P \neq S_i$ for all $1 \leq i \leq t$. Then, by Lemma 15, there exists a subset $I(t) \subseteq I$ with $|I(t)| \geq \sqrt{|I|}$ such that the linkages $\mathcal{R}(t)_i$ for $i \in I(t)$ either pairwise cross between P and S_t or pairwise do not cross. Thus setting $S_{t+1} = P$ and $\mathcal{R}(t+1)_i = \mathcal{R}(t)$ satisfies a.-d.

The other case is only slightly more complicated. Assume there exists an index x such that $P = S_x$. Consider the auxiliary graph obtained by creating a path with vertices $B(t)_i^x$ and $B(t)_i^{t+1}$ for $i \in I$, with the order of the vertices given by the order in which they occur on S_x. We add edges connecting $B(t)_i^x$ and $B(t)_i^{t+1}$ for all $i \in I$, and now we apply Lemma 14 to the auxiliary graph. We find a subset $I(t+1) \subseteq I$ with $|I(t+1)| \geq (|I|)^{\frac{1}{3}}$ such that the edges $B(t)_i^x B(t)_i^{t+1}$ for $i \in I(t+1)$ either pairwise cross, are pairwise nested, or give pairwise disjoint intervals of the underlying path. In the first two cases, there exists an edge e of S_x such that if we split S_x on e we obtain two components S^1 and S^2 such that $B(t)_i^x$ is contained in S^1 for all $i \in I(t+1)$ and $B(t)_i^{t+1}$ is contained in S^2 for all $i \in I(t+1)$. It follows that setting S_x to S^1 and S_{t+1} to be S^2, and $\mathcal{R}(t+1)_i = \mathcal{R}(t)_i$, we satisfy a.-d.

Alternatively, there for every $i \in I(t+1)$, the corresponding subpaths of the auxiliary graph connecting $B(t)_i^x$ and $B(t)_i^{t+1}$ are pairwise disjoint. In this case, we can find indices $\pi(1), \ldots, \pi(2t+1)$ and label the ends of S_i s_i and t_i such that for $B(t)_{\pi(1)}^x, B(t)_{\pi(1)}^{t+1}, B(t)_{\pi(2)}^x, B(t)_{\pi(2)}^{t+1}, \ldots, B(t)_{\pi(2t+1)}^x, B(t)_{\pi(2t+1)}^{t+1}$ occur on S_x when traversing from s_x to t_x, and for $x < i \leq t$, the paths $B(t)_{\pi(1)}^i, B(t)_{\pi(2)}^i, \ldots, B(t)_{\pi(2t+1)}^i$ occur on S_i in that order when traversing from s_i to t_i. If there exists two linkages that twist between distinct S_i and S_{i+1} for $x \leq i \leq t$, then we are able to reroute. But otherwise, we may apply Observation 5 t times, and in either case find an equivalent linkage to $\mathcal{P}(t)$ avoiding some vertex. Since $\mathcal{P}(t)$ was obtained from $\mathcal{P}$ by a series of edge splits, we conclude that there exists a linkage equivalent to $\mathcal{P}$

avoiding some vertex of $\mathcal{P}$. This completes the proof of the lemma. $\square$

We now reconsider Observation 5. Observation 5 describes how, given a linkage $\mathcal{Q}$ traversing a linkage $\mathcal{P}$ with several fairly restrictive properties, we are able to reroute the linkage $\mathcal{P}$ and avoid some vertex of the graph. The next lemma shows how the conditions *i.* and *ii.* in Observation 5 can be relaxed. The proof follows immediately from Lemma 16.

Lemma 17. *Let k and l be positive integers. There exist integers R and N such that the following holds. Let $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ be linkages each of order R and length at most l which traverse $\mathcal{P}$. Furthermore, assume there exist pairwise disjoint subpaths S_i of P containing all the endpoints of $\mathcal{Q}_i$ for $1 \leq i \leq n$. Then there exists a linkage $\mathcal{P}'$ equivalent to $\mathcal{P}$ in $\mathcal{P} \cup \bigcup_1^n \mathcal{Q}_i$ avoiding some vertex of $\mathcal{P}$.*

Proof. We let R' and N' be the values given by Lemma 16 with k, l, and $n = 2$, $r = 1$. We prove the lemma for $R = R'$ and $N = lN'$. As a slight technicality, Lemma 16 assumes all the linkages have the same length. By our choice of N, there exists a subset $I \subseteq \{1, \ldots, N\}$ with $|I| \geq N'$ such that linkages $\mathcal{Q}_i$ for $i \in I$ all have length l' for some integer $l \leq l$.

We apply Lemma 16 to the linkages $\mathcal{Q}_i$, $i \in I$. Outcome 2 in the lemma is a desired outcome of Lemma 17, thus, we may assume that there exist indices $\pi(1), \pi(2)$ and components $R_{\pi(i)} \subseteq \mathcal{Q}_{\pi(i)}$ for $i = 1, 2$ such that the linkage $R_{\pi(1)} \cup R_{\pi(2)}$ traverses $\mathcal{P}$. It follows that there exist disjoint basis subpaths B_1 and $B_{l'}$ of P such that each of $R_{\pi(i)}$ has one end in B_1 and the other end in $B_{l'}$. However, this contradicts the fact that there exist disjoint subpaths $S_{\pi(i)}$ containing both ends of $R_{\pi(i)}$. This contradiction completes the proof of the lemma. $\square$

D Finding a traversing linkage and proof of Theorem 1

In this section, we show how the proof of Theorem 1 can be reduced to Theorem 3. To do so, we will use the Weak Structure Theorem to show that a unique linkage of sufficiently large tree width contains a large traversing linkage. To apply the Weak Structure Theorem, we need the following theorem of Robertson and Seymour.

Theorem 18. *Let $\mathcal{P}$ be a linkage of order k in a graph G with $V(G) = V(\mathcal{P})$. If G contains K_{3k+1} as a minor, then there exists a vertex v and a linkage $\mathcal{P}'$ in $G - v$ equivalent to $\mathcal{P}$.*

We will need two lemmas before proceeding with proof of the Unique Linkage Theorem.

Let $C_1, \ldots, C_s$ be disjoint cycles in a plane graph G. Let D_i be the disc in the plane with boundary C_i. We say that they are *concentric* if we have the property that $D_s \subseteq \cdots \subseteq D_1$. Let G be a graph and $\mathcal{P}$ be a linkage in G. Let H be a plane subgraph of G and let $C_1, \ldots, C_s$ be concentric cycles in H. A *local peak of $\mathcal{P}$ in $C_1, \ldots, C_s$* is a subpath Q of $\mathcal{P}$ with both endpoints contained in C_i, $i > 1$, such that every internal vertex of Q contained in $\bigcup_1^s V(C_j)$ is contained in $V(C_{i-1})$. We allow the case that the subpath Q has no internal vertices contained in C_{i-1}. A related notion is that of a *simple rerouting*. The linkage $\mathcal{P}$ has a simple rerouting if there exists a cycle C_i, $i \geq 1$ and a subpath P of some component of $\mathcal{P}$ such that P has both endpoints in C_i, at least one vertex in C_{i+1}, and if there exists a subpath Q of C_i with the same endpoints as P with Q internally disjoint from $\mathcal{P}$. The name derives from the fact that the subpath P can be replaced in the linkage $\mathcal{P}$ by the path Q.

Lemma 19. *Let s be a positive integer. Let G' be a graph embedded in the plane and let $C_1, \ldots, C_s$ be concentric cycles in G'. Let G'' be another graph, with $V(G') \cap V(G'') \subseteq V(C_1)$. Assume that $G' \cup G''$ contains a linkage $\mathcal{P}$ with endpoints in $V(G'')$. Finally, let $v \in V(G')$ be a vertex contained in D_s. Then there exist concentric cycles $C'_1, \ldots, C'_s$ in G' bounding discs $D'_1, \ldots, D'_s$ with v in D'_s and a linkage $\mathcal{P}'$ equivalent to $\mathcal{P}$ such that $\mathcal{P}'$ does not have either a local peak in $C'_1, \ldots, C'_s$ nor a simple rerouting.*

Proof. Assume the lemma is false, and let G', G'', $\mathcal{P}$, and $C_1, \dots, C_s$ form a counterexample containing a minimal number of edges. To simplify the notation, we let $G = G' \cup G''$. By minimality, it follows that the graph $G = \bigcup_1^s C_i \cup \mathcal{P}$.

It immediately follows that $\mathcal{P}$ does not have a simple rerouting. The remainder of the proof will focus on showing that we do not have a local peak.

Note that no subpath $Q \subseteq \mathcal{P} \cap G'$ that is internally disjoint from $\bigcup_1^s C_i$ has both endpoints contained in C_j for some $1 < j \leq s$. Otherwise, we could reroute C_j through the path Q to find s concentric cycles in G' and contradict our choice of a counterexample containing a minimal number of edges. Note that we must reroute the cycle C_j to ensure that the vertex v is contained in the disc bounded by C_s; however, this is always possible, since we would not reroute the cycle C_s.

Assume, to reach a contradiction, that there exists an index $j > 1$ and a subpath Q in $\mathcal{P}$ such that Q is a local peak with both endpoints on C_j. Pick such a Q and j with j maximal. Assume Q is a subpath of $P \in \mathcal{P}$. Let the endpoints of Q be x and y. Lest we re-route P through C_j and find a counter-example containing fewer edges, there exists a component $P' \in \mathcal{P}$ intersecting the subpath of C_j linking x and y. By planarity, P' either contains a subpath internally disjoint from the union of the C_i with both endpoints in C_j, or P' contains a subpath forming a local peak with endpoints in C_{j-1}. Either is a contradiction to our choice of a minimal counterexample (note that in the second case, we may have to apply the same argument to P', but eventually we would get the first conclusion). This contradiction proves the lemma. $\square$

Definition 20. We define a *target* T to be a graph such that the following holds:

a. there exist disjoint cycles $C_1, \dots, C_m$ and a linkage $\mathcal{P}$ such that $T = \mathcal{P} \cup \bigcup_1^m C_i$,

b. there exists a plane subgraph H of T such that $C_1, \dots, C_s$ are concentric cycles in H, and

c. the linkage $\mathcal{P}$ does not have either a local peak in $C_1, \dots, C_s$ or a simple rerouting.

The value s is the *height* of the target. Let P be a subpath of component of $\mathcal{P} \cap D_1$ with both ends in C_1 and otherwise internally disjoint from C_1. We assume that P does not intersect D_s except for the vertices in C_s. Observe that $P \cup C_1$ forms two internally disjoint discs F_1 and F_2 with $F_1 \cup F_2 = D_1$ and $F_1 \cap F_2 \subseteq P$. We may assume F_1 contains the disc D_s. We define the *wedge of T formed by P* to be the subgraph $W = F_2 \cap H$. The *height* of the wedge W is the maximum i such that $C_i \cap W \neq \emptyset$.

Note that for any target of height s with concentric cycles $C_1, \dots, C_s$ and linkage $\mathcal{P}$, it follows immediately from the definition that for any index k, $C_{k+1}, C_{k+2}, \dots, C_s$ with the linkage $\mathcal{P}$ forms a target of height $s - k$.

Lemma 21. *For all integers l, w, k, there exists an $s = s(l, w, k)$ such that the following holds. Let T be a target of height m with concentric cycles $C_1, \dots, C_m$ and assume that the linkage $\mathcal{P}$ of order k is given. Assume $V(T) = V(\mathcal{P})$. Then either there exists a linkage $\mathcal{Q}$ traversing $\mathcal{P}$ of order w and length l, or there exists a linkage equivalent to $\mathcal{P}$ avoiding some vertex of T.*

Proof. We will prove a slightly stronger statement to facilitate an inductive proof. We will show:

Claim 22. *There exists a value $h = h(l, w, k)$ such that if W is a wedge of height h in a target T formed by a subpath P of $\mathcal{P}$, then W either contains a linkage $\mathcal{Q}$ traversing $\mathcal{P}$ of order w and length l such that P is the first basis subpath of the traversing linkage, or there exists a linkage equivalent to $\mathcal{P}$ avoiding some vertex of T.*

Proof. The proof will be by induction on l. So, we assume lexicographical order l, w, k for $h(l, w, k)$. When $l = 1$, the statement is trivial if we assume $h(1, w, k) \geq w$. We fix the values k and w, and let W and T be given. We let s be the height of the target T, and we assume $l \geq 2$.

We fix $t = h(l - 1, w', k)$ where t and w' will be later chosen in order to make the claim true. Let z be a vertex of $C_h \cap P$. If we look at the two components P_1 and P_2 of $P - z$, each intersects each of the cycles $C_1, \ldots, C_{h-1}$ in exactly one subpath by the fact that there are no local peaks by Lemma 19. We let x_i be a vertex of $P_1 \cap C_i$ for $t \leq i \leq h - 1$ and y_i a vertex of $P_2 \cap C_i$ for $t \leq i \leq h - 1$. We let Q_i be the subpath of C_i linking x_i and y_i in W.

The proof will now proceed roughly as follows. Let U_1 be the plane graph bounded by the cycle C_1 in the target T. As we traverse Q_i from x_i to y_i, by the fact that there are no simple reroutings in the target T, we know we encounter some other component P' of $\mathcal{P} \cap U_1$. This component P' forms a wedge W' contained as a subgraph of W, and by our choice of t, the wedge W' contains a traversing linkage of length $l - 1$ and order w'. Thus, if departing from x_i for many distinct indices i, we arrive at distinct wedges W', we can merge many of these linkages using Lemma 16 to find a big traversing linkage of length $l-1$ which extends to a traversing linkage of length l with P' as the first basis subpath. Alternatively, if departing from x_i for many distinct indices i, we arrive at the same wedge W', we have already begun to construct our traversing linkage with P as the first basis subpath, and P' as the second.

We give the constants we will need in the remainder of the proof. We let N and $w' = R$ be the value given by Lemma 16 applied with k, $l - 1$, $n = 2^l w$, and $r = 1$. We also assume $s \geq t + N^l + 1$ and $N \geq t$.

Let $W_1, \ldots, W_{l'}$ be wedges formed by components $P_1, \ldots, P_{l'}$ of $\mathcal{P} \cap U_1$ with $W_1 \supseteq W_2 \supseteq \cdots \supseteq W_{l'}$ and $W_1 = W$, $P_1 = P$. Moreover, there exists an index $x \geq t$ such that the following holds: for all i, $x \leq i \leq x + N^{l-l'+1}$, traversing Q_i from x_i to y_i, the first l' components of $\mathcal{P}$ encountered are exactly $P_1, P_2, \ldots, P_{l'}$ in that order. Note that such a choice clearly if $l' = 1$ because $s \geq t + N^l + 1$. We choose $W_1, \ldots, W_{l'}$, $P_1, \ldots, P_{l'}$, and x such that the value l' is maximized. Note that if $l' = l$ (and we assume $N \geq w$), then the statement is proven and there exists a traversing linkage with basis subpaths $P_1, \ldots, P_{l'}$ and order $N \geq w$.

Let x_i' be the first vertex of $P_{l'} \cap C_i$ when traversing Q_i from x_i to y_i for $x \leq i < x + N^{l-l'+1}$. Then if we consider in the subgraph $W_{l'}$, again by the fact that there are no simple reroutings in a target, we see that if we traverse Q_i from x_i' to y_i, we encounter some component of $\mathcal{P}$ before intersecting $P_{l'}$ second time. Let R_i be the first such component of $\mathcal{P} \cap U_1$ we intersect. Note that R_i forms a wedge of height at least $t = h(k, l - 1, w')$. Also, note that by planarity, we have that if $R_i \neq R_j$, then the wedges formed by R_i and R_j are disjoint, and, if $i \leq j \leq k$ and $R_i = R_k$, we have $R_j = R_i$.

If there exists a subset I of N distinct indices i, $I \subseteq \{x, \ldots, x + N^{l-l'+1} - 1\}$ such that the R_i are pairwise distinct, then by the induction hypothesis, since $N \geq t$, there exist linkages $\mathcal{L}_i$ traversing $\mathcal{P}$ for all $i \in I$, each of width $w' = R$ and length $l - 1$. We apply Lemma 16. If we find an equivalent linkage to $\mathcal{P}$ avoiding some vertex, the claim is proven. Thus, we may assume we find a subset $I' \subseteq I$ of $w2^l$ distinct indices and paths $L_i \in \mathcal{L}_i$ for all $i \in I$ such that L_i has an endpoint in R_i, and, moreover, we have the property that $\bigcup_{i \in I'} L_i$ forms a linkage traversing $\mathcal{P}$ of length $l - 1$. We may assume, in fact, that each L_i has the vertex x_i' as an endpoint. If we consider the set of paths $\overline{L}_i = L_i \cup x_i Q_i x_i'$ for $i \in I'$, we see that $\bigcup_{i \in I'} \overline{L}_i$ almost form a traversing linkage of order w and length $l - 1 + l'$, with the paths $P_1, \ldots, P_{l'}$ forming the additional basis subpaths. The only possible difficulty is that each of the P_i may intersect a basis subpath of $\bigcup_{i \in I'} L_i$. However, each of the paths L_i is disjoint from P_j for all $i \in I'$, $1 \leq j \leq l'$ because L_i is contained in the wedge formed by R_i. Thus, by possibly discarding at most half of the paths in $\bigcup_{i \in I'} L_i$ for each P_j, $1 \leq j \leq l'$, we see there exists a subset $I'' \subseteq I'$ with $|I''| \geq w$ such that $\bigcup_{i \in I''} \overline{L}_i$ forms a traversing linkage of order w and length l with P_1 a basis subpath containing an endpoint of each $\overline{L}_i$.

We may therefore assume that there does not exist such a subset I of indices and there are at most N distinct paths R_i. It follows that there exists an index x such that at least $N^{l-(l'+1)+1} + 1$ of the R_i are in fact equal to R_x. By our observations on the planarity of T, we may assume that in fact $R_i = R_x$ for all $x \leq i \leq N^{l-(l'+1)+1}$ for an index x. This contradicts our choice of l' maximal by setting $P_{l'+1} = R_x$, completing the proof of the claim.

□

The lemma now follows immediately from Claim 22 as every target of height s contains a wedge of height s. □

The next theorem essentially allows us to reduce the proof of the Unique Linkage Theorem to the proof of Theorem 3.

Theorem 23. *Let k be a positive integer, and let $\mathcal{P}$ be a linkage of order k in a graph G with $V(G) = V(\mathcal{P})$. For all integers r and l there exists an integer w such that the following holds. If the treewidth of G is at least w, then there exists a linkage $\mathcal{P}'$ equivalent to $\mathcal{P}$ such that either $\mathcal{P}'$ avoids some vertex v of G, or, there exists a linkage $\mathcal{Q}$ traversing $\mathcal{P}'$ of order r and length l.*

Proof. Fix the value k. We may assume, by Theorem 18 that G does not contain K_{3k+1} as a minor. By Theorem 13, there exists a subset A of at most $\alpha(k)$ vertices such that $G - A$ has a decomposition $G', H_0, H_1, \dots, H_n$ such that $H = \bigcup_0^n H_i$ contains a t-wall W. Note we may make t as large as necessary by increasing the value w. We choose such a decomposition to minimize n.

First, we split the linkage $\mathcal{P}$ on every edge incident the vertex set A. This results in a linkage $\mathcal{P}'$ of order at most $2\alpha + k$. To complete the proof of the theorem, it suffices to show that there exists a vertex v of G and a linkage equivalent to $\mathcal{P}'$ avoiding v by Observation 6.

The linkage $\mathcal{P}'$ may have endpoints contained in H. However, there is only bounded number of such vertices, and so we may select a subgraph H' of H containing a t'-wall W' with boundary cycle C' such that H' has a flat wall decomposition $H'_0, \dots, H'_{n'}$ satisfying the following:

1. $\partial_{G-A}(V(G) \setminus V(H')) \subseteq V(C') \cap V(H'_0)$, and
2. no endpoint of a path in $\mathcal{P}'$ is contained in H'.

Note that since the number of vertices to avoid in W is fixed (in terms of k), we may again choose t' arbitrarily large by forcing t to be large.

It follows from the fact that $V(H') \subseteq V(\mathcal{P}')$, and the fact that every vertex of H' has degree 2 in $\mathcal{P}'$, that for all $i \geq 1$, $V(H'_i)$ consists of a single induced subpath Q of some component of $\mathcal{P}'$. It follows that H'_i is planar and can be embedded with the vertices of $V(H'_i) \cap V(H'_0)$ on the boundary of the infinite face. Thus, by our choice of near embeddings to minimize the value n, we conclude that $n' = 0$ and that H' is a planar subgraph of G.

We fix an embedding of H'_0 and let $C_1, \dots, C_s$ be concentric cycles contained in the disc D_1 bounded by the cycle C_1. We may choose $s = \lfloor t'/2 \rfloor$. By Lemma 19, we may assume that there exist concentric cycles $\overline{C}_1, \dots, \overline{C}_s$ and a linkage $\overline{\mathcal{P}}$ equivalent to $\mathcal{P}'$ such that $\overline{\mathcal{P}}$ has neither a local peak in $\overline{C}_1, \dots, \overline{C}_s$ nor a simple rerouting. It follows that $\overline{\mathcal{P}} \cup \bigcup_1^s \overline{C}_i$ forms a target of height s.

If we set s to the value given by Lemma 21 for $k + 2\alpha$, l, and w, we see that there exists a linkage $\mathcal{P}'$ equivalent to $\mathcal{P}$ such that either there exists a linkage $\mathcal{Q}$ traversing $\mathcal{P}'$ of order w and length l, or $\mathcal{P}'$ avoids some vertex in G. This completes the proof of the theorem. □

The proof of Theorem 1 now follows easily, assuming Theorem 3.

Proof of Theorem 1, assuming Theorem 3. Let G be a graph and $\mathcal{P}$ a linkage of order k in G. Assume $V(G) = V(\mathcal{P})$. Let l and w be the values given in the statement of Theorem 3. Then by Theorem 23, there exists a value *width* such that if the treewidth of G is at least *width*, then there exists a linkage $\mathcal{P}'$ equivalent to $\mathcal{P}$ such that either $\mathcal{P}'$ avoids some vertex v of G, or there exists a linkage $\mathcal{Q}$ traversing $\mathcal{P}'$ of order w and length l. Theorem 3 implies that there exists a linkage $\mathcal{P}'$ equivalent to $\mathcal{P}$ in $\mathcal{P} \cup \mathcal{Q}$ avoiding some vertex of G. Thus it follows that if the tree width of G is at least *width*, then there exists a linkage $\mathcal{P}'$ equivalent to $\mathcal{P}$ avoiding some vertex of G. This proves Theorem 1. $\square$

E Switching Perspective

In this section, we outline the main steps in the argument for the proof of Theorem 3. We cover many of the same points already discussed in the main article. We include this section for completeness of the appendix.

Let k, w, and l be integers. Let $\mathcal{P}$ be a linkage of order k and $\mathcal{Q}$ be a traversing linkage of order w and length l. Let the basis subpaths of $\mathcal{Q}$ be $B_1, B_2, \ldots, B_l$.

We contract all the edges of $\mathcal{P} \cup \mathcal{Q}$ incident a vertex of degree at most 2. We also contract all the edges of $\mathcal{P} \cap \mathcal{Q}$. Thus we may assume that:

1. that there are no edges contained in $\mathcal{Q} \cap \mathcal{P}$, and
2. there are no vertices in $V(\mathcal{Q}) \setminus V(\mathcal{P})$.

Intuitively, until now we have been thinking of the linkage $\mathcal{P}$ as a sort of underlying graph, and looking at how the linkage $\mathcal{Q}$ visits each of the components in turn. We now would like to shift perspectives, and keep the linkage $\mathcal{Q}$ fixed in our minds, and look at how a component P of $\mathcal{P}$ intersects $\mathcal{Q}$ as we travel along P.

By the definition of a traversing linkage, we can fix a labeling $Q_1, Q_2, \ldots, Q_w$ of the components of $\mathcal{Q}$ so that every component $P \in \mathcal{P}$ satisfies the following. The path P can be decomposed into subpaths $R_1, \ldots, R_t$ and edges $e_1, \ldots, e_{t-1}$ that are pairwise disjoint so that e_i connects the ends of R_i and R_{i+1} and each R_i has one end in Q_1, the other end in Q_w, and intersects the paths of $\mathcal{Q}$ in order, i.e. $Q_1, Q_2, \ldots, Q_w$. (In fact, the paths R_i are the basis subpaths of $\mathcal{Q}$ on P, ordered by traversing P).

If we look at the graph formed by $P \cup \mathcal{Q}$, we see that $\mathcal{Q} \cup \bigcup_1^t R_i$ forms a subdivision of a $(w \times t)$-grid. The horizontal paths of the grid are the Q_i for $1 \leq i \leq w$, and the vertical paths are the R_i, $1 \leq i \leq t$. The edges e_i form a matching with the ends contained in $Q_1 \cup Q_w$. In the rest of the proof, we place the grid aside, and focus on these edges e_i.

The edges e_i together with the outer face boundary of W form a society. We build on the results in [30], and then extend them in such a way that the outcomes include "genus addition", i.e, a handle addition and a crosscap addition. This will be done in Section F. Using the main lemma of Section F, we adapt some ideas in [32, 33] to grow a graph on the surface with large representativity. These arguments are covered in Sections G and H. This process terminates either when we find a large clique minor (which allows us to reroute $\mathcal{P}$ to avoid some vertex by Theorem 18), or alternatively, when we embed all but a bounded number of the edges e_i in a surface of bounded genus. In this case, we use the tools for analyzing linkages developed in [29]. Let us emphasis that our proof does not need most of the heavy machinery in Graph Minor theory. Specifically, we do not use the structure theorem. We are able to avoid doing so because our society consists of only a matching, and that matching is glued onto the outside boundary of a grid. This allows us to avoid any global connectivity issues, since the structures we find are already nicely connected to the grid. And, furthermore, the society consisting of only a matching leads to considerably simpler arguments. In contrast, Graph Minor Theory needs more ingredients, including "vortex", "embedding, up to 3-separations", "tangle, respectful tangle" etc. This leads to a dramatic savings in space and effort in our proof.

Thus the remainder of the proof will now focus on these edges e_i. The first step in these arguments, which we will cover in the next few sections, will be a series of results for analyzing these edges e_i.

F An Erdős-Pósa result for societies

In this section, we return to societies which were introduced in Section B. We will prove a lemma about Ω-paths in societies. First, we give several more notions concerning societies. Let (G, Ω) be a society. A subset of vertices $X \subseteq \Omega$ is a *segment* of the society vertices Ω if there do not exist vertices $x_1, x_2 \in X$ and vertices $y_1, y_2 \in \Omega \setminus X$ such that x_1, y_2, x_2, y_2 occur in Ω in that order.

We say a society (G, Ω) is *rural* if G can be embedded in the disc with the vertices of Ω on the boundary of the disc in the order specified by Ω. We say two disjoint paths Ω-paths P_1 and P_2 *cross* if the ends of P_i can be labeled s_i and t_i for $i = 1, 2$ such that the vertices s_1, s_2, t_1, t_2 occur in Ω in that order. Equivalently, two paths cross if there do not exist disjoint segments S_1 and S_2 such that S_i contains the endpoints of P_i for $i = 1, 2$. Note that in a rural society, there do not exist disjoint crossing Ω-paths.

We extend the idea of independent bumps to societies as follows. We say that disjoint Ω-paths $P_1, \dots, P_k$ are *independent* if there exist disjoint segments S_i in Ω such that S_i contains the endpoints of P_i. Note that two disjoint Ω-paths are independent if and only if they do not cross.

Let $P_1, \dots, P_k, Q_1, \dots, Q_k$ be pairwise disjoint Ω-paths, and assume that P_i and Q_i cross. Furthermore, assume there exist nested segments $S_1, S_2, \dots, S_k$ such that the endpoints of P_i and Q_i are contained in $S_i \setminus S_{i-1}$ for all $1 \leq i \leq k$. Then the paths $P_1, \dots, P_k, Q_1, \dots, Q_k$ form k *nested crosses*.

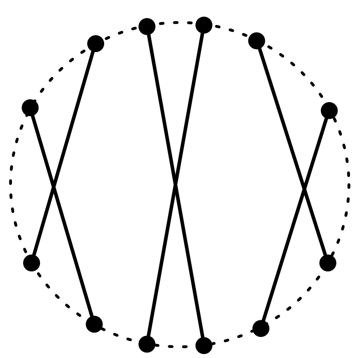

Figure 4: An example of 3-nested crosses.

We now give the main result of this section.

Lemma 24. *Let (G, Ω) be a society such that $E(G)$ is a matching. There exists a function $f(k)$ such that the following holds.*

1. *There exist k independent edges in (G, Ω).*
2. *There exists a k-crosscap.*
3. *There exists a k-handle.*
4. *There exists a set $Z \subseteq E(G)$ of size at most $f(k)$ such that $(G - Z, \Omega)$ is rural.*

5. *There exist k nested crosses.*

Before proceeding with the proof of Lemma 24, we give the following useful lemma. A *transaction* in a society (G, Ω) is a linkage $\mathcal{P}$ of Ω-paths such that there exist two disjoint segments S_1 and S_2 of Ω such that every component $P \in \mathcal{P}$ has one end in S_1 and one end in S_2. A transaction is *rural* if it does not contain a 2 disjoint crossing Ω-paths.

The following two lemmas are essentially the same as Lemmas 14 and 15, translated into terms of societies.

Lemma 25 ([30])**.** *Let $\mathcal{P}$ be a set of order k^3 of pairwise disjoint Ω-paths in a society (G, Ω). Then $\mathcal{P}$ contains a k-crosscap, k pairwise disjoint, independent Ω-paths, or a rural transaction of order k.*

Lemma 26. *Let $\mathcal{P}$ be a transaction of order k^2 in a society (G, Ω). Then either $\mathcal{P}$ contains a sublinkage forming a rural transaction of order k, or a k-crosscap.*

We now proceed with the proof of Lemma 24.

Proof. Lemma 24. We prove the lemma with $f(k) = 2(4k^2)^6$. We first observe that for all $t \geq 1$, either (G, Ω) contains vertex disjoint edges $e_1, \ldots, e_t$, $f_1, \ldots, f_t$ such that e_i and f_i cross for $i = 1, \ldots, t$, or, there exists a set Z of at most $2t$ edges such that $G - Z$ does not contain a cross. In other words, $G - Z$ is rural. Such edges $e_1, \ldots, e_t, f_1, \ldots, f_t$ may in fact be selected greedily. We pick such e_i and f_i for $1 \leq i \leq t$ for $t = (4k^2)^6$.

Apply Lemma 25 to the society consisting just of the edges $e_1, \ldots, e_t$. Lest we satisfy 1 or 2, we may assume that among the edges e_i, we find a planar transaction of order $(4k^2)^3$. We then apply Lemma 25 again to the corresponding edges f_i of the planar transaction consisting of the edges e_i, and again find a planar transaction of order $4k^2$, among the edges f_i. We thus conclude that there exists edges $e_1, \ldots, e_{4k^2}$, $f_1, \ldots, f_{4k^2}$ such that:

i. e_i and f_i cross for $1 \leq i \leq 4k^2$,

ii. the edges e_i form a planar transaction for $1 \leq i \leq 4k^2$, and

iii. the edges f_i form a planar transaction for $1 \leq i \leq 4k^2$.

Note that i. follows from the fact that we have a planar transaction of order $(4k^2)^3$ consisting of the edges e_i. There also exist segments S_1, S_2 of Ω such that every edge e_i has one end in S_1 and one end in S_2. We may assume that the endpoints of $e_1, e_2, \ldots, e_{4k^2}$ occur on S_1 in that order. Note that for all indices j, if the edge f_j crosses $e_{i(1)}$ and $e_{i(2)}$ for two values $i(1) \leq i(2)$, then f_j crosses $e_{i(3)}$ for all $i(1) \leq i(3) \leq i(2)$.

Consider the edges $f_1, \ldots, f_k$. If they all cross e_{2k+1}, then $f_1, \ldots, f_k$, $e_{k+1}, \ldots, e_{2k}$ form a k-handle. Thus we may assume that there exists an index j, $1 \leq j \leq k$ such that f_j does not cross e_{2k+1}. We now look at $f_{3k+1}, \ldots, f_{4k}$. If each one crosses e_{2k+1}, then $f_{3k+1}, \ldots, f_{4k}$ and $e_{2k+1}, \ldots, e_{3k}$ form a k-handle. We conclude there is an index j', $3k+1 \leq j' \leq 4k$ such that $f_{j'}$ does not cross e_{2k+1}. It follows that the edge e_{2k+1} divides Ω into two disjoint segments S_1 and S_2 such that one, say S_1 contains all the endpoints of e_j and f_j, and the other S_2 contains the endpoints of every edge e_i, f_i for $i \geq j'$. Since there are $4k^2$ indices for e_i, f_i, thus it follows by induction on k that (G, Ω) contains k nested crosses, which proves the lemma. □

G Extending a k-crosscap or k-handle

In Section F, one outcome when considering a huge society comprised of a matching was that we found either a large handle or crosscap. In the applications, we will want to embed the remaining edges of the matching into the faces of this handle or crosscap. The goal of this section is to characterize when we are able to do so.

Let (G, Ω) be a society with G a matching. Let M' be a k-crosscap or k-handle for some positive integer k. A *facial set of* M' is a subset $X \subseteq \Omega$ such that if we embed (M', Ω) in the natural way into the disc plus a crosscap (or in the disc plus a handle in the case that M' is a k-handle) with Ω on the boundary of the disc in the order specified by Ω, then X is the set of Ω vertices contained in a single face of the embedding.

For the lemma, we will need to allow for one more possible outcome. Let (G, Ω) be a society, and let $P_1, \dots, P_t, Q_1, \dots, Q_t$ be disjoint Ω-paths. We say that $P_1, \dots, P_t, Q_1, \dots, Q_t$ form t *twisted nested crosses* if there exists a segment S of Ω containing exactly one endpoint of P_i and Q_i for all $1 \leq i \leq t$ and if we let Ω' be the cyclic order obtained from Ω by reversing the order of the vertices of S, then $P_1, \dots, P_t, Q_1, \dots, Q_t$ form t nested crosses in (G, Ω').

The next lemma is the main result of this section.

Lemma 27. *Let t be a positive integer. There exists a function $f(t)$ such that the following holds. Let (G, Ω) be a society, and assume $E(G)$ can be partitioned into a matching M on Ω and a cycle C with $V(C) = \Omega$ such that the natural order of the cycle is the same as the cyclic order Ω. Assume that M contains a matching M' which is either a $f(t)$-crosscap or a $f(t)$-handle. Then one of the following holds*

1. *There exists a subgraph $M'' \subseteq M'$ such that M'' is either a t-crosscap or a t-handle and every edge of $G - M''$ has both endpoints in a facial set of M''.*
2. *M contains t independent bumps.*
3. *M contains t nested crosses or t twisted nested crosses.*
4. *There exists a path P in C such that if we let Ω' be the order of $\Omega - P$ given by Ω, the the following holds. In (G, Ω') there exists a set $\mathcal{P}$ of disjoint Ω'-paths such that $\mathcal{P}$ contains disjoint independent Ω'-paths $Q_1, \dots, Q_t$ such that for every $1 \leq i \leq t$, there exist t distinct components in $\mathcal{P}$ crossing Q_i.*

Proof. Assume the theorem is false, and let (G, Ω) form a counterexample. Let M', C, and M be as in the statement. We let $S_1, \dots, S_m$ be the segments of Ω demarcated by the vertices of $V(M') \cap \Omega$ so that $C[S_i]$ form internally disjoint subpaths of C. So, each $C[S_i]$ is contained in some facial set. We label the S_i such that $S_1, \dots, S_m$ occur in that order in the order Ω. If M' is a $f(t)$-crosscap, note $m = 2f(t)$ and if M' is a $f(t)$-handle, then $m = 4f(t)$. We will see in the course of the proof how large $f(t)$ must be in order to derive a contradiction.

We consider an auxiliary graph A with vertex set $S_1, \dots, S_m$ where two segments S_i and S_j are adjacent if there exists an edge e of M with one end in S_i and one end in S_j, but no facial set of M' contains both ends of e.

First, we observe that for any pair of indices i and j, there exists a subgraph M_1 of M' such that S_i and S_j are contained in the same facial set of M_1 and if M' is a $f(t)$-crosscap, then M_1 is an $f(t)/2$-crosscap, and if M' is a $f(t)$-handle, then M_1 is an $f(t)/2$-handle. We let l be a value such that

$$f(t) \geq t2^l.$$

Lest we satisfy 1, we conclude that the auxiliary graph A has at least l distinct edges.

Claim 28. *The auxiliary graph A does not have a vertex of degree $(t+4)(4t)^2$.*

Proof. Assume, to reach a contradiction, that there exists an index z such that S_z has degree $(t+4)(4t)^2$ in A. If we select every $(t+4)^{th}$ S_i adjacent to S_z while following the cyclic order Ω, it follows that we can find indices $\pi(1), \ldots, \pi\left(4t^2\right)$ with the following properties.

a. For all indices $1 \leq j \leq (4t)^2$, there exists an edge of M, label it e_j, with one end in S_z and one end in $S_{\pi(j)}$;

b. for all j, $1 \leq j \leq (4t)^2 - 1$, there exist at least t distinct edges of M' with at least one end in the segment of $\Omega \setminus (S_{\pi(j)} \cup S_{\pi(j+1)})$ avoiding S_z and no end equal to an internal vertex of S_z;

c. $\pi(1) \leq \pi(2) \leq \ldots, \leq \pi\left(4t^2\right)$.

Note that b. follows from the fact that M' is either a $f(t)$-crosscap or a $f(t)$-handle. By Lemma 26, it follows that the set of edges $e_1, \ldots, e_{(4t)^2}$ contains either a rural transaction of order $4t$ or a $4t$-crosscap. By pairing sequential edges in this rural transaction or crosscap, we see that there exist $2t$ disjoint independent Ω'-paths $P_1, \ldots, P_{2t}$ with ends in $\Omega' = \Omega \setminus S_z$. Moreover, for each Ω'-path P_i, there exist t distinct edges of M' each with exactly one end in the Ω'-segment linking the ends of P_i and the other end in Ω'. Since M' has at most two independent edges, it follows that at least $2t-2$ of the Ω'-paths P_i are each crossed by t distinct edges of M'. We conclude that 4. is satisfied, a contradiction. This completes the proof of the claim. □

We have shown that the auxiliary graph A does not have a vertex of large degree. The remainder of the proof will consist in showing that A does not have a large matching.

Claim 29. *If A contains a matching of order at least $9(18t^2(t+2))^3$, then there exist edges e_i, f_i, and segments T_i for $1 \leq i \leq t^2(t+2)$ with the following properties.*

a. The segments T_i are disjoint and occur in the order $T_1, \ldots, T_{t^2(t+2)}$ in Ω.

b. The edges e_i and f_i each have exactly one end in T_i and are each disjoint from T_j for all $1 \leq i, j \leq t^2(t_2)$, $j \neq i$.

c. The edges e_i, $1 \leq i \leq t^2(t+2)$, either form a crosscap or a rural transaction; the edges f_i, $1 \leq i \leq t^2(t+2)$, either form a crosscap or a rural transaction.

d. If the edges e_i form a crosscap, then e_i and f_i do not cross for all $1 \leq i \leq t^2(t+2)$; if the edges e_i do not pairwise cross, then e_i and f_i do cross for all $1 \leq i \leq t^2(t+2)$.

e. There exists a segment T containing T_i, and exactly one endpoint of each edge e_i and f_i for all $1 \leq i \leq t^2(t+2)$.

Proof. For every edge in the matching in A, let an edge f_i be an edge of $M \setminus M'$ whose endpoints are not contained in a single facial set. We set T_i to be a segment S_j containing an endpoint of f_i. Note that since M is a matching, the endpoint of f_i in T_i is an internal vertex of the segment. Consider the two edges of M' incident the first and last vertices of T_i. If M' is a crosscap, since f_i does not have both ends in the facial set of the crosscap containing T_i, at least one of the edges is not crossed by f_i. We set e_i to be such an edge. If M' is a handle, then f_i must cross one of the edges of M' incident T_i. In this case, let e_i be such an edge.

Each T_i intersects T_j for at most two indices j. Thus, by discarding at most two thirds of the indices i, we may assume that the T_i are pairwise disjoint. After possibly re-ordering the indices, we conclude that there exist e_i, f_i, and T_i, $1 \leq i \leq 3(18t^2(t+2))^3$ satisfying the following. For $1 \leq i \leq 3(18t^2(t+2))^3$, the edges e_i and f_i have exactly one endpoint in T_i. The segments T_i, $1 \leq i \leq 3(18t^2(t+2))^3$ are pairwise disjoint and occur in that order in Ω. The edges e_i, $1 \leq i \leq 3(18t^2(t+2))^3$ either form a crosscap or are a subset of the edges of handle. Finally, the edges e_i and f_i, for $1 \leq i \leq 3(18t^2(t+2))^3$, satisfy property d.

The proof will proceed by discarding e_i, f_i, and T_i for some subset of the indices $1 \leq i \leq 3(18t^2(t+2))^3$ to find a set satisfying all the properties a - e. We see already that we satisfy property a. and d. Again, by the property that each edge of $M \setminus M'$ has each endpoint contained as an internal vertex of the segment S_i for some index i, we see that for all $1 \leq j \leq 3(18t^2(t+2))^3$, the edge e_j intersects at most one T_l for $l \neq j$, and the edge f_j intersects at most one T_l for $l \neq j$. We conclude that there exists a subset $I \subseteq \{1, \ldots, 3(18t^2(t+2))^3\}$ with $|I| \geq (18t^2(t+2))^3$ such that e_i, f_i, and T_i for $i \in I$ satisfy a, b, and d.

We apply Lemma 25 to the set of edges f_i, $i \in I$. There do not exist t independent edges in (G, Ω), so we conclude that there exists a set $I_1 \subseteq I$ such that f_i, $i \in I_1$ is either a $|I_1|$-crosscap or a rural transaction. Moreover, $|I_1| \geq |I|^{\frac{1}{3}} \geq 18t^2(t+2)$. If the edges e_i are not in the crosscap, by construction they are a subset of the handle. Thus, there exists $I_2 \subseteq I_1$, $|I_2| \geq |I_1|/2 \geq 9t^2(t+2)$ such that the edges e_i for $i \in I_2$ form either a crosscap or a rural transaction. We conclude that e_i, f_i, and T_i, for $i \in I_2$ satisfy a - d.

In the final step, we must show the existence of the segment T. Let T' be the minimal segment containing T_i for all $i \in I_2$. If the edges e_i form the rural transaction, then there exist at most two indices i and i' such that e_i and $e_{i'}$ are independent and have both their endpoints in T'. It follows that T' can be partitioned into 3 segments such that no edge e_i has both endpoints in one such segment. If the edges e_i form a crosscap, then there do not exist two indices i and i' such that e_i and $e_{i'}$ have both their endpoints in T' and are independent. In this case, T' can be partitioned into two segments such that no edge e_i has both endpoints in one of the segments. We conclude that there exists a subset $I_3 \subseteq I_2$ with $|I_3| \geq |I_2|/3$ such that e_i, f_i, and T_i satisfy a-d and there exists a segment T'' containing T_i for all $i \in I_3$, and no edge e_i has both endpoints in T''. Re-iterating the same argument for the edges f_i, we see that after again discarding possibly 2/3 of the indices, we find a subset $I_4 \subseteq I_3$, $|I_4| \geq t^2(t+2)$ such that a-e holds for e_i, f_i, and T_i, $i \in I_4$. This completes the proof of the claim. □

We will show that given such e_i, f_i, and T_i as in Claim 29, either 3 or 4 in the statement of the lemma is satisfied. There are now four cases to consider: the edges e_i either form a crosscap or a rural transaction, and the edges f_i either form a crosscap or a rural transaction. However, two of these cases are essentially symmetric to the others. The next claim is a slight strengthening of outcomes 3 and 4 in order to take advantage of this symmetry.

Claim 30. *Let e_i, f_i, T_i, and T satisfy a. − e. in Claim 29 for $1 \leq i \leq t^2(t+2)$. Then either (G, Ω) satisfies 3 with the additional property that every edge of the nested crosses or twisted nested crosses has one end in T and one end in $\Omega \setminus T$, or (G, Ω) satisfies 4 with the additional property that the path P satisfies $V(P) \subseteq \Omega \setminus T$.*

Proof. We first define the idea of a flip on a segment. If S is a segment of Ω, the cyclic order Ω' is obtained by a *flip on S* if Ω' is an order on the same set of vertices as Ω obtained by following S in the same order as in Ω, followed by the reverse order on $\Omega \setminus S$. We observe that (G, Ω) satisfies 3 or 4 with the additional properties in the statement if and only if (G, Ω') does as well where Ω' is the cyclic ordered obtained by a flip on T. Thus, it suffices to prove the claim up to flips on T.

By possibly flipping on T, we may assume that the edges e_i do not cross. It follows that there are two possible cases - whether the edges f_i cross or not. We analyze each case separately.

Case: The edges f_i pairwise cross. Consider the edge $e_{t(t+2)+1}$, and let z be the endpoint of $e_{t(t+2)+1}$ in $\Omega \setminus T$. The vertex z partitions $\Omega \setminus (T \cup z)$ into two segments, say F_1 and F_2, so that F_1, z, F_2, T occur in Ω in that order. If $f_{t(t+2)+1}$ has an endpoint in F_1, then f_i has an endpoint in F_1

for every $i \le t(t+2)$. If we link $f_{i(t+2)+1}$ and $f_{(i+1)(t+2)}$ in F_1, we conclude that outcome 4 is satisfied with $P \subseteq F_1$. Similarly, if $f_{t(t+2)+1}$ has an end in F_2, we satisfy outcome 4 with P contained in F_2. This completes the analysis of the first case.

Case: The edges f_i pairwise do not cross. First, we consider what happens if there exists an index x such that f_x crosses $e_{x+t(t+2)}$. It follows that f_x crosses e_i for all $x \le i \le x+t(t+2)$. Moreover, since the edges f_i do not cross, we see that f_i crosses e_j for all $x \le i \le x+t(t+1)$ and $i \le j \le x+t(t+2)$. We conclude that $\Omega \setminus T$ can be partitioned into two segments F_1 and F_2 such that F_1 contains the ends of f_i for $x \le i \le x+t(t+2)$ and F_2 contains the endpoints of e_i for $x \le i \le t(t+2)$. By linking every $f_{x+i(t+2)}$ and $f_{x+(i+1)(t+2)-1}$ in F_1 for $0 \le i \le t-1$, we see that conclusion 4 in the lemma is satisfied with the additional property that the path P is disjoint from T, as desired. We conclude that no such index x exists. It follows that the crossing edges $e_{it(t+2)+1}$, $f_{it(t+2)+1}$ for $0 \le i \le t-1$ form t nested crosses, as desired. This completes the analysis of the second case.

The analysis of these two cases completes the proof of the claim. □

Claims 29 and 30 together imply that the auxiliary graph does not contain a matching of order $9(18t^2(t+2))^3$. If we also consider Claim 28, we see that the auxiliary graph A has at most $\left(2(t+4)(4t)^2\right) 9(18t^2(t+2))^3$ edges. However, by choosing the function f to be large, we can guarantee that the auxiliary graph have arbitrarily many edges. This final contradiction completes the proof of the lemma. □

H Growing a surface

In this section, we put together the results from the previous sections to arrive at one of the main lemmas we will use in the proof of Theorem 3.

We will need the next theorem. Robertson and Seymour showed a complete characterization of when a given society (G,Ω) is rural. In the case that $V(G)$ is just the society vertices, the statement and proof are considerably more simple.

Theorem 31 (Robertson and Seymour [30]). *Let (G,Ω) be a society with $\Omega = V(G)$. Then (G,Ω) is rural if and only if there do not exist two crossing edges.*

Let (G,Ω) be a society with G consisting of a matching. Let M' be a k-crosscap or k-handle. If X is a facial set of M', we define Ω_X the *facial order* to be the cyclic ordering of the vertices of X given by traversing the facial cycle when we embed (M',Ω) in the disc plus a crosscap or in the disc plus a handle, respectively.

Lemma 32. *Let t be a positive integer. There exists a function $f(t)$ such that the following holds. Let (G,Ω) be a society, and assume $E(G)$ can be partitioned into a matching M on Ω and a cycle C with $V(C) = \Omega$ such that the natural cyclic order given by the cycle is the same as the cyclic order Ω. Then one of the following holds*

1. *There exists a subgraph M_1 of M such that M_1 is either a k-crosscap (or a k-handle) such that that every edge of $M - M_1$ is contained in a facial set of M_1. Moreover, for all but possibly one facial set X, if we let Ω_X be the facial order of the vertices of X and H_X the edges of M with both endpoints in X, then (H_X,Ω_X) is rural.*
2. *There exists a set $Z \subseteq M$ of size at most $f(t)$ such that $M - Z$ is rural.*
3. *There exist t independent bumps in M.*

4. *There exist t nested crosses or t twisted nested crosses.*

5. *There exists a path P in C such that if we let Ω' be the order of $\Omega - P$ given by Ω with the following property. In (G, Ω') there exists a set $\mathcal{P}$ of disjoint Ω'-paths such that $\mathcal{P}$ contains disjoint independent Ω'-paths $Q_1, \dots, Q_t$ such that for every $1 \leq i \leq t$, there exist t distinct components in $\mathcal{P}$ crossing Q_i.*

Proof. Let f_1 be the function given in the statement of Lemma 24. We let f_2 be the function in the statement of Lemma 27. Finally, we let f be equal to

$$f(k) = f_1(f_2(k2^{3k+1})).$$

To simplify the notation, we fix $\alpha = k2^{3k+1}$. Lemma 24 implies that either we can find a set Z of at most $f(k)$ edges such that $(G - Z, \Omega)$ is rural, or we satisfy outcome 2 or 3 above, or there exists a subgraph M' equal to a handle or a crosscap of order $f_2(\alpha)$. We apply Lemma 27 to the subgraph M'. Again, either we satisfy one of the conclusions above, or there exists a subgraph M'' equal to an α-handle or an α-crosscap such that every edge of $M \setminus M''$ has both endpoints in some facial set of M''.

Again, we observe that for any two facial sets X and Y of M'', there exists an $\alpha/2$-crosscap or handle $M''' \subseteq M''$ such that X and Y are contained in the same facial set X' of M'''. Moreover, the facial sets of M''' not equal to X' are the same facial sets as in M''.

Let m be a positive integer and let $X_1, \dots, X_m$ be the facial sets of M''. Let H_i be the subgraphs of M given by the edges with both ends in X_i for $1 \leq i \leq m$, and let Ω_i be the corresponding facial order of X_i. If there exist $3k+1$ distinct indices i such that (H_i, Ω_i) is rural, it follows that we satisfy outcome 1 in the lemma by applying the above remark $3k+1$ times (then we would get either a k-handle or a k-crosscap).

Thus we may assume that there exist such $3k+1$ such indices for which (H_i, Ω_i) is not rural. Note that there exists at most one facial set X_i of M'' which cannot be partitioned into two segments of Ω_i. Thus, we see that there exist $3k$ distinct indices i for which (H_i, Ω_i) is not rural and Ω_i can be partitioned into two segments S_i^1 and S_i^2 such that each S_i^j is simply connected in a topological sense in Ω for $j = 1, 2$.

As (H_i, Ω_i) is not rural and consists of edges in Ω plus a matching, it follows from Theorem 31 that if (H_i, Ω_i) contains a pair e_i and f_i of crossing edges. If there exists k distinct indices i such that either e_i or f_i has both endpoints contained in a single segment S_i^j for $j = 1$ or 2, then we see that (G, Ω) satisfies outcome 3. It follows that for $2k$ indices i, each of the edges e_i and f_i have exactly one endpoint in S_i^1 and the other endpoint in S_i^2. We now see that if M'' is a crosscap that the edges e_i and f_i form a k twisted nested crosses in (G, Ω). Otherwise, M'' is a handle, and we see that half of the pairs of edges e_i and f_i form k nested crosses in (G, Ω). This completes the proof of the lemma. □

I Repeatedly crossed independent paths

As an outcome of Lemmas 27 we find a large number of independent Ω-paths, each of which is crossed by many disjoint Ω-paths. In the application to our matching society attached to the outside boundary of a large grid, these Ω-paths may possibly use part of the boundary cycle. In this case, the structure of the linkage $\mathcal{P}$ is lost. However, such Ω-paths will allow us to find a large clique minor which will allow us to apply Theorem 18. The next lemma is the main result of this section.

Recall that the *$(l \times m)$-grid* to be the graph with vertex set $v(i, j)$ for $1 \leq i \leq m$ and $1 \leq j \leq l$ such that $v(i, j)$ and $v(i', j')$ are adjacent if $|i - i'| + |j - j'| = 1$. The indices i, j form the normal cartesian coordinates of the vertices of the grid. The outer cycle of the grid to be the cycle bounding the infinite face give the standard embedding of the grid in the plane.

Lemma 33. *For all $t \geq 1$, there exists integers k, l, N such that the following holds. Let H_1 be an $(n \times m)$-grid with $n, m \geq N$. Let Ω be the branch vertices in the outer cycle of H_1 in the natural cyclic order given by the outer cycle, and let (H_2, Ω) be a society with H_2 a matching. If (H_2, Ω) contains k disjoint independent Ω-paths $e_1, \ldots, e_k$ such that for each e_i there exist l distinct edges that cross e_i, then $H_1 \cup H_2$ contains K_t as a minor.*

Until this point, we have not worked explicitly with minors and have been able to use the necessary graph minor results as a sort of "black box". However, in the proof of Lemma 33 we will need to explicitly find a large clique minor in the graph. We remind the reader that a graph G contains K_t as a minor if and only if there exist disjoint sets of vertices $X_1, \ldots, X_t$ such that each induces a connected subgraph of G, and for every pair of indices i, j, there exists an edge of G with one end in X_i and one end in X_j. The sets X_i are called the *branch sets* of the minor.

In the proof of Lemma 33, we will eventually reduce to one of several possible cases. We first show that in each such case, we can find a large clique minor. Consider the following graph. Let H^1_r be the graph formed by taking an $2r \times 2r$-grid and labeling the vertices $v(i,j)$ with the natural cartesian coordinates, and adding edges of the form $v(i,r)v(i+1,r+1)$ and $v(i,r+1)v(i+1,r)$ for $1 \leq i \leq 2r-1$. In other words, H_{2r} is constructed by taking a $(2r \times 2r)$-grid and adding a pair of crossing edges in each face of the middle row of faces.

Lemma 34. *Let $t \geq 1$ be an integer. The graph $H^1_{4t^2}$ contains K_t as a minor.*

Proof. The proof is by induction on t. We actually prove a slightly stronger statement, to facilitate the induction. We show that $H^1_{4t^2}$ contains a K_t minor such that every branch set contains a vertex in the top row of the grid, i.e. a vertex equal to $v(i, 4t^2)$ for some value i. The statement is clearly true for $t \leq 4$.

We now assume $t > 4$, and let $H^1_{4t^2}$ be given. Observe that the set of vertices $\{v(i,j) : 1 \leq i \leq 4(t-1)^2, 2(2t+1)+1 \leq j \leq 4t^2 - 2(2t+1)\}$ induces a subgraph isomorphic to $H^1_{4(t-1)^2}$, and so by induction contains the branch sets $B_1, \ldots, B_{t-1}$ of a K_{t-1} minor such that B_i contains a vertex of the form $v(i, 4t^2 - 2(2t+1))$ for some index i, $i \leq 4(t-1)^2$. We let B_t be the set of vertices

$$\begin{aligned} B_t = &\{v(i, 4t^2) : 1 \leq i \leq 4(t-1)^2 + 2t\} \\ &\cup \{v(4(t-1)^2 + 2t, i) : 2t^2 \leq i \leq 4t^2\} \\ &\cup \{v(4(t-1)^2 + 2t + j, 2t^2) : 2 \leq j \leq 2t,\ j \text{ even}\} \\ &\cup \{v(4(t-1)^2 + 2t + j, 2t^2 + 1) : 1 \leq j \leq 2t-1,\ j \text{ odd}\} \end{aligned}$$

Then B_t induces a connected subgraph of $H^1_{4t^2}$. Moreover, there exist $t-1$ disjoint paths from each of the B_i, $1 \leq i \leq t-1$ to the set of vertices $\{v(i, 4t^2) : 4(t-1)^2 + 2t + 1 \leq i \leq 4t^2\}$. Each of these disjoint paths will use one crossing edge of the crossing edges in the middle strip and will consequently have an edge to B_t. We conclude that $B_1, \ldots, B_t$ form the branch sets of a K_t minor satisfying the stronger hypothesis, completing the proof of the lemma. □

Let H^2_r be constructed as follows. We begin with a $(2r \times 4r^3)$-grid with the vertices labeled $v(i,j)$ for $1 \leq i \leq 4r^3$ and $1 \leq j \leq 2r$. Moreover, we add the edges e^i_j for $1 \leq i \leq r$, $1 \leq j \leq 2r^2$ where the endpoints of e^i_j are $v(2r(j-1)+i, 2r)$ and $v(2r(j-1)+i+r, 2r)$. The graph H^2_r can be thought of, then, as a long grid with $2r^2$ distinct r-crosscaps glued to the top of the grid in a series.

Lemma 35. *Let $t \geq 1$ be a positive integer. Then H^2_t contains K_t as a minor.*

Proof. Fix the integer t, and let H_t^2 be given. Consider the graph $\overline{H}^2$ defined as follows. We begin with a $2t \times 2t$-grid, to which we add the edges e_i defined as follows. For $1 \leq i \leq t$, the endpoints of e_i are $v(i, 2t)$ and $v(i+t, 2t)$. The graph H_t^2 can be decomposed into $2t^2$ copies of $\overline{H}^2$ glued together in a path-like manner.

To prove the lemma, we will find paths $P_1, \dots, P_t$ in H_t^2 such that the endpoints of P_i are $v(1, i)$ and $v(4t^3, t-i+1)$ such that for any pair of indices i and j, there exists an edge of H_t^2 with one end in P_i and one end in P_j. The paths $P_1, \dots, P_t$ will then comprise the branch sets of a K_t minor.

Fix an integer k, $k \leq t$. In the graph $\overline{H}^2$, there exists paths $P'_1, \dots, P'_t$ such that the endpoints of P'_i are $v(1, i)$ and $v(2t, i)$ for $i < k$ and $v(1, i)$ and $v(2t, t-i)$ for $i \geq k$. Note that there is an edge connecting P'_t and P'_{k-1}. Consequently, in H_t^2 restricted to the vertices $\{v(i, j) : 1 \leq i \leq 4t\}$, there exist paths P''_i, $1 \leq i \leq t$ with the endpoints of P''_i equal to $v(1, i)$ and $v(4t, i)$ with the property that there is an edge between P''_t and P''_{k-1}. Thus, in H_t^2 restricted to the vertices $\{v(i, j) : 1 \leq i \leq 4t(t-1), 1 \leq j \leq 2t\}$, there exist paths linking $v(1, i)$ to $v(4t(t-1), i)$ such that the $v(1, t)$ path has an edge to every other path.

We can now extend the path P''_t along the bottom edge of the grid in H_t^2 to a path terminating at $v(4t^3, 1)$, and inductively find the desired paths from $v(4t(t-1), i)$ to $v(4t^3, t-i+1)$ for $2 \leq i \leq t$. This completes the proof of the lemma. □

The final special graph we will have to consider is as following. Let H_r^3 be defined as follows. We begin with a $((r+1)r^3 \times 2r)$-grid with the vertices labeled $v(i, j)$ for $1 \leq i \leq (r+1)r^3$ and $1 \leq j \leq 2r$. As above, we add additional edges e_j^i for $1 \leq i \leq r+1$, $1 \leq j \leq r^3/2$, but now the endpoints of e_j^i are of two different kinds. For $i = 1$ and $1 \leq j \leq r^3/2$, we have that the endpoints of e_j^i are equal to $v(2(r+1)(j-1)+1, 2r)$ and $v(2(r+1)(j-1)+r+1, 2r)$. For $2 \leq i \leq r+1$ and $1 \leq j \leq r^3/2$, the endpoints of e_j^i are equal to $v(2(r+1)(j-1)+i, 2r)$ and $v(2(r+1)(j)+1-i, 2r)$.

Lemma 36. *Let $t \geq 1$ be a positive integer. Then H_t^3 contains K_t as a minor.*

Proof. Fix $t \geq 1$, and let H_t^3 be given. As in the proof of Lemma 36, we consider the following subgraph of H_t^3. Let $\overline{H}^3$ be constructed from the $(2(t+1) \times 2t)$-grid by adding the edges $v(1, 2t)v(t+1, 2t)$ and $v(i, 2t)v(2(t+1)+2-i, 2t)$. The graph H_t^3 can be thought of as $t^3/2$ copies of $\overline{H}^3$ glued together in a path-like structure.

To construct a K_t minor, we find paths $P_1, \dots, P_t$ from $\{v(1, 1), \dots, v(1, t)\}$ to $\{v((t+1)t^3, 1), \dots, v((t+1)t^3, t)\}$ such that for every pair of indices i and j, there exists an edge with one end in P_i and the other end in P_j. First, observe that for any choice of j, $j < t$, in $\overline{H^3}$ there exist paths $P'_1, \dots, P'_t$ such that the endpoints of P_i are equal to $v(1, i)$ and $v(2(t+1), i)$ for $i \leq j$, and equal to $v(1, i)$ and $v(2(t+1), i+1)$ for $j < i < w$ and $v(1, w)$ and $v(2(t+1), j+1)$ for $i = j$. It follows that in the subgraph of H_t^3 restricted to the vertices $\{v(i, j) : 1 \leq i \leq [t(t-1)/2]\, 2(t+1), 1 \leq j \leq 2(t+1)\}$ there exist disjoint paths P''_i, $1 \leq i \leq t$ such that the endpoints of P''_i are equal to $v(1, i)$ and $[v(t(t-1)/2]\, 2(t+1), i)$ and moreover, for all $i < t$, there exists an edge with one end in P''_i and P''_t. We use one more copy of $\overline{H^3}$ and route P''_t to a path using the bottom row of the grid, and inductively find the desired $t-1$ paths. This completes the proof of the lemma. □

We will also use the following easy combinatorial lemma. It shows that given a collection of subsets of some ordered set Ω, then if the sets are sufficiently large, we can sacrifice some of the elements and separate the sets by intervals in Ω.

Lemma 37. *Let l, k be positive integers. Let Ω be an ordered set of elements, and let $S_1, \dots, S_l$ be subsets of Ω. Assume $|S_i| \geq kl^2$ for all $1 \leq i \leq l$. Then there exists subsets $S'_i \subseteq S_i$ and intervals $I_i \subseteq \Omega$ for all $1 \leq i \leq l$ such that $I_i \cap I_j = \emptyset$ for all $i \neq j$, $S'_i \subseteq I_i$, and $|S'_i| \geq k$ for all $1 \leq i \leq l$.*

Proof. The proof is by induction on l. Clearly, the statement holds when $l = 1$. We now assume $l \geq 2$. Note that we may also assume that every element of Ω is contained in some set S_i for some index i. Let I be the interval consisting of the first kl elements. There exists some index i such that $|S_i \cap I| \geq k$. Moreover, for every $j \neq i$, S_j has at least $kl^2 - kl \geq k(l-1)^2$ elements not contained in I. Thus, we fix $I_i = I$ and apply the induction hypothesis to the sets $S_j - I$, $j \neq i$ contained in the sets $\Omega - I$. $\square$

We now proceed with the proof of Lemma 33.

Proof. Fix $t \geq 1$. Let (H_2, Ω) and H_1 be as stated, and let $e_1, \dots, e_k$ be disjoint independent edges in H_2. We let $f_1, \dots, f_s$ be the edges of $H_2 \setminus \{e_1, \dots, e_k\}$. Let I_i for $1 \leq i \leq k$ be the subset of indices j, $1 \leq j \leq s$ such that f_j crosses e_i. Note that by assumption, $|I_i| \geq l$ for all $1 \leq i \leq k$. Finally, for each edge e_i, $1 \leq i \leq k$, we fix the segment S_i of Ω containing the endpoints of e_i so that for all $j \neq j'$, $S_j \cap S_{j'} = \emptyset$.

We will prove that $H_1 \cup H_2$ contains one of $H^1_{4t^2}$, H^2_t, or H^3_t. Then Lemmas 34, 35, 36 will complete the proof. We set $k = 2(4(8t^3)^3)^2$ and $l = 2(4(8t^3)^3)^2(4t^4)$. We set $N = k + l$.

We would like to be able to assume that the sets I_i are pairwise disjoint, however, this certainly will not be the case in general. By discarding some of the edges e_i and some of the edges f_i, we will be able to guarantee this property.

Claim 38. *There exists a subset $J^1 \subseteq \{1, \dots, k\}$, and subsets $I^1_j \subseteq I_j$ for $j \in J^1$ such that $|J^1| \geq k/2$ and $|I^1_j| \geq l/2$ such that the following holds. For all $j \in J^1$ and for all $i \in I^1_j$, the edge f_j does not have an endpoint in $S_{j'}$ for all $j' \neq j$.*

Proof. We consider a weighted auxiliary graph defined as follows. The vertex set is $\{1, \dots, k\} \cup x$, where x is a dummy vertex. Two vertices i and i' are adjacent if there exists an edge f_j with endpoints in both S_i and $S_{i'}$. The vertex x is adjacent to i if there exists an edge f_j with one end in S_i and one end in $\Omega \setminus \bigcup_1^s S_{i'}$. The weight of an edge is number of distinct edges f_j with endpoints in the corresponding sets.

A classic result of Erdős says that in any weighted graph, there exists a partition of the vertices such that every vertex has at least half of the total weight of its incident edges with the opposite end in the other side of the partition. It follows that there exists a partition (X, Y) of the vertices of our auxiliary graph so that for every $i \in X$, at least half of the total weight of it's incident edges have an end in Y. One of X or Y has at least half the indices i, say X, and we let $J^1 = \{i : i \in X\}$. For all $i \in J^1$, we let I^1_i be the edges f_i with ends correspond to vertices in opposite sides of the partition (X, Y). This proves the claim. $\square$

Let J^1 and I^1_j, $j \in J^1$ be as in the claim. We have the property that $|J^1| \geq (4(8t^3)^3)^2$ and $|I^1_j| \geq (4(8t^3)^3)^2(4t^4)$ for all j. For all $j \in J^1$, let $X_j \subseteq \Omega$ be the vertices of $\Omega - S_j$ equal to an end an edge f_i for some $i \in I_j$. We apply Lemma 37 to the sets X_j, $j \in J^1$, to find $I^2_j \subseteq I^1_j$ and segments $T_j \subseteq \Omega$ for all $j \in J^1$ such that:

i. $T_j \cap T_{j'} = \emptyset$ for all $j \neq j'$.

ii. For all $j \in J^1$ and for all $i \in I^2_j$, the edge f_i has one end in S_j and one end in T_j.

iii. For all $j \in J^1$, $|I^2_j| \geq \sqrt{\frac{2(4(8t^3)^3)^2(4t^4)}{2(4(8t^3)^3)^2}} \geq 2t^2$.

As a quick technicality, it is possible that $S_j \cap T_j \neq \emptyset$. However, observe that by possibly discarding half the indices in each I^2_j, we may assume that $T_j \cap S_j = \emptyset$ for all j. Finally, by applying Lemma 26, for every $j \in J^1$, we may pick $I_j^3 \subseteq I_j^2$ such that the edges f_i, $i \in I_j^3$ either form a k-crosscap or a rural transaction of order k.

We want the property that the $T_j \cap S_{j'} = \emptyset$ for all j, j'.

Claim 39. *There exists a subset $J^2 \subseteq J^1$ such that for all $j, j' \in J^2$, we have that $S_j \cap T_{j'} = \emptyset$, and we may choose J^2 with $|J^2| \geq \frac{1}{2}\sqrt{|J^1|} \geq (8t^3)^3$.*

Proof. Fix an index $j \in J^1$. If there exist $\frac{1}{2}\sqrt{|J^1|} + 2$ distinct indices j' with $T_{j'} \cap S_j \neq \emptyset$, then $J^2 = \{j' \in J^1 : T_{j'} \subseteq S_j\}$ satisfies the claim because $T_j \cap T_{j'} = \emptyset$. Thus, we may assume that at most $\frac{1}{2}\sqrt{|J^1|} + 1$ distinct $T_{j'}$ intersect S_j. Similarly, at most $\frac{1}{2}\sqrt{|J^1|} + 1$ distinct $S_{j'}$ intersect T_j. Thus at most $\sqrt{|J^1|} + 2$ indices have their corresponding segments intersect $S_j \cup T_j$. We may then greedily select a set of indices $J^2 \subseteq J^1$ of size $\frac{1}{2}\sqrt{|J^1|}$ satisfying the claim. $\square$

Finally, we apply Lemma 25 to J^2. One outcome is to find disjoint segments $R_1, \ldots, R_{8t^3}$ each containing both S_j and T_j for some index $j \in J^2$. In this case, we see that $H_1 \cup H_2$ contains either the graph H_t^2 or H_t^3 as a minor, and consequently, a K_t minor by Lemma 35 or 36, respectively. Alternatively, there exist two segments R_1^* and R_2^*, such that R_1^* contains $4t^3$ distinct S_j and R_2^* contains the corresponding T_j. In this case, we take two disjoint $(2(4t^2) \times 2(4t^2))$-grid minors, one containing many of the S_j in R_1^* and one containing many of the T_j in R_2^*. In this case, the graph $H_1 \cup H_2$ contains $H_{4t^2}^1$ as a minor, and consequently also contains K_t as a minor, which proves Lemma 33. $\square$

J The proof of Theorem 3

We prove in this section that given a linkage $\mathcal{P}$ and a sufficiently large and long traversing linkage $\mathcal{Q}$, that $\mathcal{P}$ is not unique in $\mathcal{P} \cup \mathcal{Q}$. Assume Theorem 3 is false. We fix a value k and let $\mathcal{P}$ be a linkage of order k and let $\mathcal{Q}$ be a traversing linkage of $\mathcal{P}$ of length $l = l(k)$ and order $w = w(k)$ contradicting the statement of Theorem 3. Assume that from all such contradictory linkages, we choose $\mathcal{P}$ and $\mathcal{Q}$ to minimize the number of vertices in their union $\mathcal{P} \cup \mathcal{Q}$. In the course of the proof we will see exactly how large l and w must be in order to arrive at a contradiction.

First, we observe that by our choice to minimize the number of vertices, that there are no vertices of degree two or one in $\mathcal{P} \cup \mathcal{Q}$, otherwise we could contract an edge and find a smaller counter-example to the theorem. Similarly, there do not exist any edges contained in $\mathcal{P} \cap \mathcal{Q}$.

We return to the perspective discussed in Section E. We label the components of $\mathcal{Q}$ as $Q_1, Q_2, \ldots, Q_w$ such that every component P of $\mathcal{P}$ can be partitioned into subpaths $R_1, \ldots, R_t$ and edges $e_1, \ldots, e_{t-1}$ for some positive t where the endpoints of e_i are equal to an endpoint of R_i and and endpoint R_{i+1}. Moreover, we may select the paths $R_1, \ldots, R_t$ to satisfy the property that each R_i intersects the each of the paths $Q_1, \ldots, Q_w$ and in that order. The paths R_i are in fact the basis subpaths of $\mathcal{Q}$ which happen to be subpaths of P.

Notice that union of $\mathcal{Q}$ as well as all the paths $R_1, \ldots, R_t$ for every component $P \in \mathcal{P}$ form a $(w \times l)$-grid. Label this grid subgraph W. As additional notation, we fix M to be the set of these edges $e_1, \ldots, e_t$ for every path $P \in \mathcal{P}$. Then M is a matching such that every edge has it's endpoints in $Q_1 \cup Q_w$. Note that every vertex of $Q_1 \cup Q_w$ is the end of some edge in M.

If we let Ω be the natural cyclic ordering of $V(Q_1) \cup V(Q_w)$ by following Q_1 and then Q_w in clockwise order around the boundary cycle of W.

We first show the following.

Claim 40. *There exists a value K_1 (depending only on k) such that (M, Ω) does not contain K_1 independent Ω-paths.*

Proof. Assume, to reach a contradiction, that there exist edges $e_1, \ldots, e_{K_1}$ forming K_1 independent Ω-paths. We will see in the proof how big K_1 must be in order to make the statement true. Let $S_1, \ldots, S_{K_1}$ be the pairwise disjoint segments of Ω containing the endpoints of $e_1, \ldots, e_{K_1}$, respectively. We choose such edges e_i and segments S_i to minimize $\bigcup_{i=1}^{K_1} S_i$.

Observe that by our choice to minimize $\bigcup_{i=1}^{K_1} S_i$, it follows that for every i and every vertex v of $S_i - e_i$, there exists some edge of M with one end equal to v and the other end in $\Omega \setminus S_i$. We know that the graph $\mathcal{P} \cup \mathcal{Q}$ does not contain K_{3k+1} as a minor by Theorem 18. Let k', l', N' be the integers obtained from Lemma 33 by excluding K_{3k+1} as a minor. It follows that there exists a set I of at least $K_1 - k'$ distinct indices such that for all $i \in I$, we have the property that $S_i - e_i$ has at most l' vertices. Note that we are assuming that both l and w are at least N'.

For every $i \in I$, we have that $|S_i| \leq l' + 2$. As a slight technicality, we discard possibly two edges e_i in order to assume that e_i has both ends either contained in Q_1 or both ends contained in Q_w. As a further technicality, we discard possibly half the indices in I so that we may assume that every S_i is contained in the same path, either in Q_1 or in Q_w. Let $\mathcal{Q}_i$ be the sublinkage of $\mathcal{Q}$ restricted to the basis subpaths between the two endpoints of e_i. It follows that there exists a component $P \in \mathcal{P}$ such that at least

$$\frac{K_1 - k' - 2}{2k}$$

distinct indices of I, we have that $\mathcal{Q}_i$ has all it's endpoints contained in P. Moreover, by construction, there exist disjoint segments S_i of P such that S_i contains the endpoints of $\mathcal{Q}_i$. If we let f be the function in Lemma 17 and assume that

$$\frac{K_1 - k' - 2}{2k} \geq f(k, l' + 2)$$

then Lemma 17 contradicts our assumptions on $\mathcal{P}$. Note that here we are assuming that $w \geq l'+k+4$. $\square$

The next claim is an immediate consequence of Lemma 36, and we omit the proof here.

Claim 41. *There exists a constant K_2, depending only on k, such that (M, Ω) does not contain K_2 nested crosses nor contain K_2 twisted nested crosses.*

Proceeding with the proof of the theorem, we will repeatedly apply Lemma 32. In each application, we will use Claim 40 to exclude the possibility of outcome 3, Claim 41 to exclude outcome 4, and Lemma 33 to ensure that we do not have outcome 5.

In each application of Lemma 32, we will add either a large crosscap or a handle. We use a more technical inductive hypothesis to ensure that we also maintain high representativity. Thus, either we grow to a large genus surface with high representativity, and consequently show that $\mathcal{P} \cup \mathcal{Q}$ contains a large clique minor, a contradiction, or the process terminates and we delete a bounded number of edges of $\mathcal{P} \cup \mathcal{Q}$ and embed the graph in a surface of bounded genus. In this case, we derive a contradiction via Theorem 10.

In order to maintain representativity, we will need to maintain a rooted circular grid of large depth. This leads us to a somewhat technical inductive hypothesis. The function $g(i)$ will be a decreasing function and we will see later how large it must be later in order to make the theorem true.

i. $V(H_i) = V(\mathcal{P} \cup \mathcal{Q})$ and W is a subgraph of H_i.

ii. H_i is embedded in a surface Σ_i of Euler genus $\geq i$.

iii. If Σ_i is not the sphere, then the embedding of H_i in Σ has representativity at least $g(i)$.

iv. There is a unique face F_i of H_i such that $\mathcal{P} \cup \mathcal{Q} - E(H_i)$ forms a matching with every edge having both endpoints in F_i.

v. The boundary of F_i is a cycle in H_i and can be decomposed into at most $4i$ segments of $Q_1 \cup Q_w$ of Ω.

vi. H_i has a rooted circular grid on the face F of depth $g(i)$.

We begin with H_0 equal to the grid W formed by $\mathcal{Q}$ as well as the union of every basis subpath R_i of $\mathcal{Q}$ embedded in the surface Σ_0 equal to the plane. As a slight technicality, in order to satisfy *v.*, we add two dummy edges d_1 and d_2 to the graph with the endpoints of d_1 equal to the first vertices of Q_1 and Q_w and the endpoints of d_2 equal to the last vertices of Q_1 and Q_w.

We now pick H_i and Σ_i and an embedding satisfying *i-vi* with the value i to be maximal so that the graph K_{3k+1} does not embed in the surface Σ_i. We let Ω_i be the cyclic order of the vertices of F_i given by the boundary cycle C_i of F_i. We apply lemma 32 to the graph $(C_i \cup ((\mathcal{P} \cup \mathcal{Q}) \setminus H_i), \Omega)$ with

$$t = \max\{K_1 + 4g(i), K_2 + 4g(i), k' + 1, l' + 1, N'\}.$$

where k', l', and N' are the values obtained from applying Lemma 33 for a K_{3k+1} minor. We analyze the five possible outcomes of the application of the lemma.

The easiest outcomes to consider are 3 and 4. If we have outcome 3, then since the facial boundary of F_i can be decomposed into at most $4g(i)$ segments of Ω, if there exist $K_1 + 4g(i)$ independent Ω_i edges, then at least K_1 of these edges must have their endpoint in a single segment of Ω. This contradicts Claim 40. If we have outcome 4, then by a similar argument, either there exists a single segment of Ω with K_2 nested crosses or K_2 nested twisted crosses, or there exists two segments S_1 and S_2 with the K_2 nested crosses or twisted crosses having the property that each edge has one end in S_1 and one end in S_2. These edges will then form either K_2-nested twisted crosses or K_2-nested crosses in Ω, depending on the order order of the segment S_1 and S_2 in Ω. This contradicts Claim 41.

If outcome 5 in the application of Lemma 32 holds, we proceed as follows. Let W' be the circular grid obtained by deleting the cycle F. Let $\mathcal{P}$ be the linkage obtained in Lemma 32. Then the components of $\mathcal{P}$ extend to a linkage $\overline{\mathcal{P}}$ with endpoints in the inner boundary of W'. By possibly discarding one of the paths in $\overline{\mathcal{P}}$, we find as a minor a grid W'' of depth at least $g(i) - 1$ and width at least t and a linkage $\overline{\mathcal{P}}$ allowing us to apply Lemma 33 and find K_{3k+1} as a minor, a contradiction. Note, here we are assuming that $g(i) \geq N' + 1$.

The final two cases to consider are when we have outcome 1 or 2. If we satisfy outcome 1, let $\overline{M}$ be the graph forming the t-crosscap or t-handle, as well as the edges of every rural face of the t-crosscap or t-handle. It follows from Observation 8 that $W_i \cup \overline{M}$ embeds in a surface Σ' of genus strictly greater than the order of genus than the surface Σ. Moreover, the edges not included in the new embedding have their endpoints in a single face F', and clearly there exists a circular grid of depth $g(i)/2$ rooted at F'. Also, the representativity of the embedding is at least $g(i)/2$ by the existence of the above circular grid of depth $g(i)/2$ rooted at F'. Finally, it is easy to see that if F_i can be decomposed into m segments of Ω, then the boundary cycle of the face F' can be decomposed into at most $m + 4$ segments of Ω. If we assume

$$g(i) \geq 2g(i + 1),$$

we conclude that the embedding of $H_i \cup \overline{M}$ satisfies $i - vi$.

By our choice of i, it follows that K_{3k+1} must embed in the surface Σ'. If we assume that the value $g(i)$ is sufficiently large to allow us to apply Theorem 9 for the surface Σ' and K_{3k+1} minors, we conclude that G contains K_{3k+1} as a minor, a contradiction.

The final case to consider, then, is that we satisfy outcome 2. It follows that there exists a set Z of $f(t)$ edges such that $G - Z$ embeds in a surface of genus bounded by a function of k. If we consider the linkage $\mathcal{P}'$ obtained by splitting the linkage $\mathcal{P}$ on the edges of Z, we have a linkage of order at most $k + f(t)$. We let w' be the value obtained from Theorem 10 applied to linkages of order $k + f(t)$ in a surface Σ. Theorem 10 now provides a contradiction, as we see that $G - Z$ contains a $(w' \times w')$-grid, and consequently, the tree width of $G - Z$ is at least w'. Thus the linkage $\mathcal{P}'$ can be rerouted to avoid some vertex of G, and consequently, the linkage $\mathcal{P}$ can also be rerouted to avoid some vertex of G by Observation 6. This completes the proof of Theorem 3.