A Simple Proof of Second Order Strong Normalization with Permutative Conversions

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Abstract

A simple and complete proof of strong normalization for first and second order intuitionistic natural deduction including disjunction, first-order existence and permutative conversions is given. The paper follows Tait-Girard approach via computability predicates (reducibility) and saturated sets. Strong normalization is first established for a set of conversions of a new kind, then deduced for the standard conversions. Difficulties arising for disjunction are resolved using a new logic where disjunction is restricted to atomic formulas.

Key words: strong normalization, permutative conversions, second order natural deduction

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1 Introduction

Strong normalization theorem for a logical system (and a given set of rewriting rules or reductions of derivations) states that every sequence of reductions terminates in a normal form. This kind of results play a fundamental role in proof theory and theoretical computer science [1,7]. One of the most important characteristics of a normal form is a subformula property: a normal derivation of a formula A contains only subformulas of A. Although the first normalization (cut-elimination) proof has been given by G. Gentzen for sequent calculus (L-systems), the focus of the study of normalization moved after [14] to natural deduction, for example [2,3,6,9,10,12,15]. The applications included analysis of cut elimination, analysis of proofs including consistency, effective disjunction and existence properties, and extraction of programs from proofs. Many of them rely on Curry-Howard isomorphism [8] between natural deductions and suitable extensions of lambda calculus.

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Most of the strong normalization proofs in the literature are given for the so-called negative fragment allowing &,→, ∀,¬, but not ∨, ∃. Even when ∨,∃ are included, the set of reductions most often does not include "remote" conversions performed when an introduction rule is separated from elimination rule for the same formula (such introduction/elimination pair is called a cut) by a series of ∨,∃-rules. For example, the following figure at the left is to be converted into the figure at the right:

$$\frac{\dfrac{\Gamma\vdash\exists x C[x]\qquad \dfrac{C[c],\Gamma\vdash A\quad C[c],\Gamma\vdash B}{C[c],\Gamma\vdash A\,\&\,B}}{\Gamma\vdash A\,\&\,B}}{\Gamma\vdash A}\qquad\qquad \frac{\Gamma\vdash\exists x C[x]\quad C[c],\Gamma\vdash A}{\Gamma\vdash A}$$

To include remote conversions and preserve a local character of reductions one introduces permutative conversions moving the elimination rule up the derivation until it meets an introduction component of the cut (or another obstacle). A permutative conversion transforms the left hand side into the figure

$$\frac{\Gamma\vdash\exists x C[x]\qquad \dfrac{\dfrac{C[c],\Gamma\vdash A\quad C[c],\Gamma\vdash B}{C[c],\Gamma\vdash A\,\&\,B}}{C[c],\Gamma\vdash A}}{\Gamma\vdash A}$$

then to

$$\frac{\Gamma\vdash\exists x C[x]\quad C[c],\Gamma\vdash A}{\Gamma\vdash A}$$

as intended. Leaving out permutative conversions forfeits subformula property, since a proof normal with respect to all other conversions can still contain "remote" cuts.

Our goal in this paper is to give a simple and complete proof of strong normalization for the first and second order logic including disjunction, first-order existence and permutative conversions for first order connectives, suitable both for extensions to stronger systems and for teaching. The attention in this paper is restricted to intuitionistic systems. We follow Tait-Girard approach [5,6,16] using computability predicates and saturated sets as the range for second order quantifiers. In this approach strong normalization is deduced from a stronger property, called computability (reducibility in this paper). Instead of natural deductions deriving a formula A one works (via Curry-Howard isomorphism) with corresponding λ-terms of type A. A useful device introduced in [16], called saturated sets in this paper, is to switch to untyped λ-terms, some of which can belong to a set $\overline{A}$ of computable terms of type A. The definition of $\overline{A}$ in [6,16] is given by a simple induction on the construction of A. A similar simple induction on A proves that all terms in $\overline{A}$ are strongly normalizing. After this it turned out to be possible to prove that every typed term t of type A belongs to $\overline{A}$, hence strongly normalizes. The second proof is by induction on the term t. This nice modularity is violated in most of the familiar extensions of proofs by [6,16] to reductions which include permutative conversions. In [10,15,18] the definition of the relation $t\in\overline{A}$ uses an inductive construction (inside of induction on A) which can be proved to equivalent to an explicit arithmetic definition. This makes proofs of basic properties of sets $\overline{A}$ and of $t \in \overline{A}$ longer and more difficult to follow. An alternative treatment in [2,9] also defines relation playing the role similar to $t \in \overline{A}$ by induction on t and normalization ordinal of t. [12] requires complicated inductive definitions of saturated sets.

Our proof reestablishes modularity mentioned above. Most of the obvious attempts to define sets $\overline{\exists xA}$ and $\overline{A \vee B}$ of computable terms of types $\exists xA$ and $A \vee B$ stall because the complexity of the conclusion C of the $\exists$- or $\vee$-elimination rule

$$\frac{\Gamma \vdash \exists xA \quad A[b], \Gamma \vdash C}{\Gamma \vdash C} \quad \frac{\Gamma \vdash M : A \vee B \quad u : A, \Gamma \vdash N : C \quad v : B, \Gamma \vdash L : C}{\Gamma \vdash (M, N, L)_{u,v} : C}$$

is not connected in any way with the complexity of $\exists xA$ or $A \vee B$. This difficulty is resolved in the present paper by additional conversions $(\exists), (\vee 1), (\vee 2)$ in Section 3, which allow to define $\overline{\exists xA}$ in terms of $\overline{A}$. For disjunction we use a translation of $A \vee B$ by $\exists x((q_0 x \vee q_1 x) \,\&\, ((q_0 x \to A) \,\&\, (q_1 x \to B)))$, where q_0 and q_1 are predicate constants used to distinguish components of a direct sum and satisfied essentially only by constants 0 and 1. This translation allows to restrict disjunction to atomic formulas and give particularly a simple and perspicuous set of axioms for q_0, q_1 allowing to avoid most of the difficulties other axiomatizations cause for disjunction. In particular, we define the saturated set $\overline{F \vee G}$ for atomic formulas F, G explicitly (instead of reducing it to $\overline{F}, \overline{G}$) as the property SN of being strongly normalizable. We wanted to define $M \in \overline{A \vee B}$ so that if $Mp_0 = 0$ then $Mp_1 \in \overline{A}$ and if $Mp_0 = 1$ then $Mp_1 \in \overline{B}$, but it is not a correct definition for arbitrary formulas A, B, since Mp_0 is not a constant in general. However, if $\overline{A} = \overline{B}$, the relation $M \in \overline{A \vee B}$ can be defined as $Mp_1 \in \overline{A}$. In particular, for atomic formulas F, G, the relation $M \in \overline{F \vee G}$ can be defined as $M \in SN$. We were able to get away with such an oversimplification because $\overline{F} = \overline{G} = SN$. This latter equivalence uses independence of the sets $\overline{A}$ from the first order arguments of the predicate symbols: $\overline{A[x := s]} = \overline{A}$. This property was often used implicitly in previous normalization proofs. Its explicit use simplifies many of the definitions and proofs. Another simplification results from a bold use of untyped terms. While the conversion $(\exists)$ in Section 3 agrees with traditional realizability interpretations, the conversions $(\vee 1), (\vee 2)$, strictly speaking, do not respect realizability or Brouwer-Heyting-Kolmogorov interpretation which requires for $\vee$-elimination something like "conditional conversions": $(M, N, L)_{u,v} \to$ if $Mp_0 = 0$ then $N[u := Mp_1]$, $(M, N, L)_{u,v} \to$ if $Mp_0 = 1$ then $L[u := Mp_1]$. A possibility to drop the guarding conditions for Mp_0 is a piece of luck. Because of $(\vee 1)$ and $(\vee 2)$, if the system allowed non-atomic disjunctions, the system would fail to be strongly normalizing.

Strong normalization of second order natural deduction with disjunction and existential quantification and permutative conversions is established in [12,15]. [15] needs some supplementary details in order to be complete. [12] gave in detail only a proof for second-order universal quantification and stated it can be extended to second-order existential quantification. We do not know of any other attempts of strong normalization proofs for second-order systems with permutative conversions in the literature.

There are many works on strong normalization of weaker systems. Strong normalization of second order natural deduction with disjunction and existential quantification without permutative conversions is proved in [6]. Strong normalization of second order natural deduction with existential quantification without permutative conversions is proved in [5]. Strong normalization of the negative fragment of second order natural deduction is discussed in [4,7,11,18,19]. Strong normalization of first order natural deduction with disjunction and existential quantification and permutative conversions is proved in [10,18,3]. Strong normalization of first order natural deduction with disjunction and permutative conversions is proved in [2,9]. Strong normalization of first order natural deduction with general elimination rules is discussed in [9]. Strong normalization of first order propositional $\lambda\mu$-calculus with permutative conversions is proved in [2].

An important (for the second author) pointer to a possible fruitful direction came from [17] where a partial normalization result (without $\vee/\vee$ conversions) is established for an extension of intuitionistic logic explicitly using Curry-Howard terms. Another important incentive (for the same author) came from approach suggested in [13] using a natural deduction system NJi for first order intuitionistic predicate logic. Standard elimination rules for $\exists, \vee$ are replaced in NJi by "instantiation" rules

$$\frac{\Gamma \vdash \exists x A}{<\Gamma>_{\exists x A, b} \vdash A[x/b]}\ \exists i \qquad \frac{\Gamma \vdash A_0 \vee A_1}{<\Gamma>_{A_0 \vee A_1, A_j} \vdash A_j}\ \vee i \quad (j = 0, 1)$$

It was conjectured there that a direct Tait-Girard style proof of strong normalization for NJi is feasible and would lead to a perspicuous proof for NJ with permutative conversions via the translation from NJ to NJi. The proof presented below can be seen as a confirmation of this conjecture.

Section 2 describes the definition of the second order natural deduction $NJ\forall^2$ with permutative conversions and states the strong normalization theorem. The second order logic $\mathbf{LAD_2}$ with atomic disjunction is defined in Section 3. Section 4 proves its strong normalization. In Section 5, simulation of full disjunction in $\mathbf{LAD_2}$ is discussed and a translation from $NJ\forall^2$ into $\mathbf{LAD_2}$ is given. Using this translation and the strong normalization of $\mathbf{LAD_2}$ the strong normalization of $NJ\forall^2$ is proved in that section.

2 Reductions for the second order natural deduction system $NJ\forall^2$

In this paper, we call the second order intuitionistic natural deduction with permutative conversions without second-order existential quantification the system $NJ\forall^2$. It has disjunction, first-order existential quantification and their permutative conversions. We will give the definition of the system $NJ\forall^2$.

Below we give the list of axioms and inference rules for $NJ\forall^2$ together with a standard assignment of the second order λ-terms to deductions in $NJ\forall^2$ by Curry-Howard isomorphism. The system of reductions is also standard and includes permutative conversions for $\vee, \exists$.

Definition 2.1 (Language) First order variables $x, y, z, \ldots$,

Function symbols $f, g, h, \ldots$,

Predicate variables $X, Y, \ldots$,

Predicate symbols $q, r, \ldots$,

First order terms $t, s, \ldots ::= x | f\vec{t}$,

Formulas $A, B, \ldots ::= \bot | q\vec{t} | X\vec{t} | A \to B | A \,\&\, B | \forall x A | A \vee B | \exists x A | \forall X A$,

Abstraction terms $T ::= X | \lambda \vec{x}.A$,

Term variables $u^A, v^B, w^C, \ldots$ where for every u^A and v^B, if u is v then A is B.

Definition 2.2 (Substitution) Substitution $t[x := s]$,$A[x := s]$ is defined in a familiar way.

Substitution $A_X[T]$ is defined by induction on A as follows:

$(X\vec{t})_X[Y] \equiv Y\vec{t}$

$(X\vec{t})_X[\lambda\vec{x}.A] \equiv A[\vec{x} := \vec{t}]$

$(A \,\&\, B)_X[T] \equiv (A_X[T] \,\&\, B_X[T])$

The other cases are defined similarly to the case $A \,\&\, B$.

Definition 2.3 (Terms and typing) Assumptions

$u^A : A$

Inference rules

$$\dfrac{\begin{matrix}[u^A : A] \\ \vdots \\ M : B\end{matrix}}{\lambda u^A.M : A \to B}\,(\to I) \qquad \dfrac{M : A \to B \quad N : A}{MN : B}\,(\to E)$$

$$\dfrac{M : A \quad N : B}{\langle M, N\rangle : A \,\&\, B}\,(\& I) \qquad \dfrac{M : A \,\&\, B}{Mp_0 : A}\,(\& E1) \qquad \dfrac{M : A \,\&\, B}{Mp_1 : B}\,(\& E2)$$

$$\dfrac{M : A}{\lambda x.M : \forall x A}\,(\forall I) \qquad \dfrac{M : \forall x A}{Ms : A[x := s]}\,(\forall E)$$

$$\dfrac{M : A}{\langle 0, M\rangle^{A \vee B} : A \vee B}\,(\vee I1) \qquad \dfrac{M : B}{\langle 1, M\rangle^{A \vee B} : A \vee B}\,(\vee I2)$$

$$\dfrac{M : A \vee B \quad \begin{matrix}[u^A : A] \\ \vdots \\ N : C\end{matrix} \quad \begin{matrix}[v^B : B] \\ \vdots \\ L : C\end{matrix}}{(M, N, L)_{u^A, v^B} : C}\,(\vee E) \qquad \dfrac{M : \bot}{Mp_A : A}\,(\bot E)$$

$$\dfrac{M : A[x := s]}{\langle s, M\rangle^{\exists x A} : \exists x A}\,(\exists I) \qquad \dfrac{M : \exists x A \quad \begin{matrix}[u^A : A] \\ \vdots \\ N : C\end{matrix}}{(M, N)_{x, u^A} : C}\,(\exists E)$$

$$\dfrac{M : A}{\lambda X.M : \forall X A}\,(\forall^2 I) \qquad \dfrac{M : \forall X A}{MT : A_X[T]}\,(\forall^2 E)$$

The rules $(\forall I)$, $(\exists E)$, and $(\forall^2 I)$ have a standard proviso for variables.

Type superscripts in u^A,$\langle 0, M\rangle^{A\vee B}$, $\langle 1, M\rangle^{A\vee B}$, and $\langle s, M\rangle^{\exists x A}$ are sometimes omitted to save notation.

Substitutions $M[x := N]$, $M[u^A := N]$ and $M[X := N]$ are defined in a standard way.

Definition 2.4 (Reductions) λ conversions:

(β) $(\lambda\alpha.M)R \to M[\alpha := R]$ (α is x,u^A,or X. R is s, N, or T.)

$(pair1)$ $\langle M, N\rangle p_0 \to M$

$(pair2)$ $\langle M, N\rangle p_1 \to N$

$(\vee 1)$ $(\langle 0, M\rangle, N, L)_{u^A, v^B} \to N[u^A := M]$

$(\vee 2)$ $(\langle 1, M\rangle, N, L)_{u^A, v^B} \to L[v^B := M]$

$(\exists)$ $(\langle s, M\rangle, N)_{x,u^A} \to N[x := s, u^A := M]$

Permutative conversions:

$(perm\exists)$ $(M, N)_{x,u^A} R \to (M, NR)_{x,u^A}$ (R is s,N,T,p_0, p_1, or p_C)

$(perm\exists\exists)$ $((M, N)_{x,u^A}, L)_{y,v^B} \to (M, (N, L)_{y,v^B})_{x,u^A}$

$(perm\exists\vee)$ $((M, N)_{x,u^A}, L_1, L_2)_{u_1^B, u_2^C} \to (M, (N, L_1, L_2)_{u_1^B, u_2^C})_{x,u^A}$

$(perm\vee)$ $(M, N, L)_{u^A, v^B} R \to (M, NR, LR)_{u^A, v^B}$
(R is s,N,T,p_0, p_1, or p_C)

$(perm \vee \exists)$ $((M, N, L)_{u_1^A, u_2^B}, K)_{y,v^C} \to$
$(M, (N, K)_{y,v^C}, (L, K)_{y,v^C})_{u_1^A, u_2^B}$

$(perm \vee \vee)$ $((M, N_1, N_2)_{v_1^A, v_2^B}, K_1, K_2)_{u_1^C, u_2^D} \to$
$(M, (N_1, K_1, K_2)_{u_1^C, u_2^D}, (N_2, K_1, K_2)_{u_1^C, u_2^D})_{v_1^A, v_2^B}$

Congruence.

$(congr)$ $M \to M'$

if $N \to N'$ holds and M' is obtained from M by replacing just one occurrence of N by N'.

We say that M reduces to N if $M \to N$.

Remark. Subject reduction property and Church Rosser property hold.

A term M is strongly normalizable if there is no infinite reduction sequence

$M \to M_1 \to M_2 \to \dots$

beginning with M.

Theorem 2.5 (Strong normalization) *Every term of the system $NJ\forall^2$ is strongly normalizable.*

Note that every term of $NJ\forall^2$ is typable. The rest of this paper contains a proof of this theorem.

3 System $\mathbf{LAD_2}$

The system $\mathbf{LAD_2}$ is obtained from the system $NJ\forall^2$ by the following modifications:

(1) Disjunction $F \vee G$ is allowed only for atomic formulas F, G.

(2) The reduction rules for disjunction and existential quantification are more general than in $NJ\forall^2$. We have the reduction rules $(\vee 1)$ of $(M, N, L)_{u,v} \to N[u := Mp_1]$ and $(\vee 2)$ of $(M, N, L)_{u,v} \to L[v := Mp_1]$ for disjunction where p_0 and p_1 are the projections. These rules allow us to reduce the disjunction elimination term $(M, N, L)_{u,v}$ even if M is not a disjunction introduction term $\langle 0, M' \rangle$ or $\langle 1, M' \rangle$. We have the reduction rules $(\exists)$ of $(M, N)_{x,u} \to N[x := Mp_0, u := Mp_1]$ for first-order quantification. It reduces the existential quantification elimination term $(M, N)_{x,u}$ even if M is not an existential quantification introduction term $\langle s, M' \rangle$.

(3) The reduction rules are applied to quasi-terms according to the generalized reductions so that the result of reduction includes the expressions like Mp_0. The set of quasi-terms is closed under reductions, while the set of terms is not closed under reductions.

(4) The inference rules in Definition 3.2 below are untyped versions of the corresponding rules of $NJ\forall^2$ plus the axioms c_0, c_1, c_2 for "cases" operation.

Definition 3.1 (Language) First order variables $x, y, z, \ldots$,

Function symbols $f, g, h, \ldots$ including specific constants $0, 1$,

Predicate variables $X, Y, \ldots$,

Predicate symbols $q, r, \ldots$ including specific predicate symbols q_0, q_1.

Each of function symbols, predicate variables and predicate symbols has a fixed arity. Notation X^n indicates that the predicate variable X has arity n.

Term variables $u, v, w, \ldots$,

Term constants c_0, c_1, c_2,

First order terms $t, s, \ldots ::= x | f\vec{t}$,

Atomic formulas $F, G, \ldots ::= q\vec{t}$,

Formulas $A, B, \ldots ::= \bot | q\vec{t} | X\vec{t} | A \to B | A \,\&\, B | \forall x A | F \vee G | \exists x A | \forall X A$.

A formula is also called a type.

Abstraction terms $T ::= X | \lambda\vec{x}.A$,

Each abstraction term has a fixed arity.

Substitution $A[x := s]$ is defined as usual.

Substitution $A_X[T]$ is defined in the same way as in the system $NJ\forall^2$.

Definition 3.2 (Terms and typing) Terms $M, N, L, \ldots$

Assumptions

$u : A$

Inference rules

$$\frac{\begin{array}{c}[u : A] \\ \vdots \\ M : B\end{array}}{\lambda u.M : A \to B}\ (\to I) \qquad \frac{M : A \to B \quad N : A}{MN : B}\ (\to E)$$

$$\frac{M : A \quad N : B}{\langle M, N \rangle : A \,\&\, B}\ (\& I) \qquad \frac{M : A \,\&\, B}{Mp_0 : A}\ (\& E1) \qquad \frac{M : A \,\&\, B}{Mp_1 : B}\ (\& E2)$$

$$\frac{M : A}{\lambda x.M : \forall x A}\ (\forall I) \qquad \frac{M : \forall x A}{Ms : A[x := s]}\ (\forall E)$$

$$\frac{M : F}{\langle 0, M\rangle : F \vee G}\ (\vee I1) \qquad \frac{M : G}{\langle 1, M\rangle : F \vee G}\ (\vee I2)$$

$$\frac{M : F \vee G \quad \begin{array}{c}[u : F]\\ \vdots \\ N : C\end{array} \quad \begin{array}{c}[v : G]\\ \vdots \\ L : C\end{array}}{(M, N, L)_{u,v} : C}\ (\vee E)$$

$$\frac{M : A[x := s]}{\langle s, M\rangle : \exists x A}\ (\exists I) \qquad \frac{M : \exists x A \quad \begin{array}{c}[u : A]\\ \vdots \\ N : C\end{array}}{(M, N)_{x,u} : C}\ (\exists E)$$

$$\frac{M : \bot}{Mp_\bot : A}\ (\bot E) \qquad \frac{}{c_0 : q_0 0} \qquad \frac{}{c_1 : q_1 1} \qquad \frac{}{c_2 : q_0 t \to q_1 t \to \bot}$$

$$\frac{M : A}{\lambda X.M : \forall X A}\ (\forall^2 I) \qquad \frac{M : \forall X A}{MT : A_X[T]}\ (\forall^2 E)$$

The rules $(\forall I)$, $(\exists E)$, and $(\forall^2 I)$ have a standard proviso for variables.

$\vec{N} : \vec{B} \vdash M : A$ means that there is a proof of $M : A$ from assumptions $N_1 : B_1, \ldots, N_n : B_n$ where $\vec{N} \equiv N_1, \ldots, N_n$ and $\vec{B} \equiv B_1, \ldots, B_n$.

Definition 3.3 (Quasi-term)

quasi-terms $M ::=$

$0|1|c_0|c_1|c_2|p_0|p_1|p_\bot|x|u|\lambda u.M|MM|\langle M, M\rangle|\lambda x.M|$

$(M, M)_{x,u}|(M, M, M)_{u,u}|\lambda X.M|$

$\bot|q|X|M \to M|M \,\&\, M|\forall x M|M \vee M|\exists x M|\forall X M$

From now on, $M, N, L, \ldots, P, Q, R, \ldots$ denote quasi-terms.

Note that $X\vec{t}$ and $q\vec{t}$ are quasi-terms because $M\vec{N}$ is a quasi-term if M and $\vec{N}$ are quasi-terms.

Notation. $MM_1M_2 \ldots M_n \equiv ((\ldots((MM_1)M_2)\ldots)M_n)$. It is also abbreviated by $M\vec{M}$ where $\vec{M} \equiv M_1, \ldots, M_n$.

Substitution $M[x := N]$, $M[u := N]$, $M[X := N]$ is defined in a standard way.

$\lambda u.M$ binds u in M. $\lambda x.M$, $\forall x M$ and $\exists x M$ bind x in M. $\lambda X.M$ and $\forall X M$ bind X in M. $(M, N)_{x,u}$ binds x and u in N. $(M, N, L)_{u,v}$ binds u in N and v in L. The set $FV(M)$ of the free variables of M is defined in a standard way.

Definition 3.4 (Reductions) For quasi-terms M, N, the relation $M \to N$ is defined as follows.

λ conversions.

(β) $\quad (\lambda\alpha.M)N \to M[\alpha := N]$ $\quad$ (α is x,u or X)

$(pair1)$ $\quad \langle M, N\rangle p_0 \to M$

$(pair2)$ $\quad \langle M, N\rangle p_1 \to N$

$(\vee 1)$ $\quad (M, N, L)_{u,v} \to N[u := Mp_1]$

$(\vee 2)$ $\quad (M, N, L)_{u,v} \to L[v := Mp_1]$

$(\exists)$ $\quad (M, N)_{x,u} \to N[x := Mp_0, u := Mp_1]$

Permutative conversions.

$(perm\exists)$ $(M, N)_{x,u}R \to (M, NR)_{x,u}$

$(perm\exists\exists)$ $((M, N)_{x,u}, R)_{y,v} \to (M, (N, R)_{y,v})_{x,u}$

$(perm\exists\vee)$ $((M, N)_{x,u}, R_1, R_2)_{u_1,u_2} \to (M, (N, R_1, R_2)_{u_1,u_2})_{x,u}$

$(perm\vee)$ $(M, N, L)_{u,v}R \to (M, NR, LR)_{u,v}$

$(perm \vee \exists)$ $((M, N, L)_{u_1,u_2}, R)_{y,v} \to (M, (N, R)_{y,v}, (L, R)_{y,v})_{u_1,u_2}$

$(perm \vee \vee)$ $((M, N_1, N_2)_{v_1,v_2}, R_1, R_2)_{u_1,u_2} \to$

$(M, (N_1, R_1, R_2)_{u_1,u_2}, (N_2, R_1, R_2)_{u_1,u_2})_{v_1,v_2}$

Congruence.

$(congr)$ $M \to M'$

if $N \to N'$ holds and M' is obtained from M by replacing just one occurrence of N by N'.

We say M reduces to N if $M \to N$. The relation $\to^+$ is defined as the transitive closure of the relation $\to$ and the relation $\to^*$ is defined as the reflexive transitive closure of the relation $\to$.

Remark. (1) The reduction $(\vee 1)$ is applicable even if $M = \langle 1, N\rangle$.

(2) The set of quasi-terms is closed under reduction. This is easily proved by induction on a quasi-term.

(3) Neither subject reduction property nor Church Rosser property holds. A counterexample to subject reduction property is $v : B, w : A \vdash (\langle 1, v\rangle, u_1, w)_{u_1,u_2} : A$, $(\langle 1, v\rangle, u_1, w)_{u_1,u_2} \to \langle 1, v\rangle p_1$ by $(\vee 1)$, but $v : B, w : A \not\vdash \langle 1, v\rangle p_1 : A$. A counterexample to Church Rosser property is $(v, c_0, c_1) \to c_0$ by $(\vee 1)$ and $(v, c_0, c_1) \to c_1$ by $(\vee 2)$.

Definition 3.5 A quasi-term M of $\mathbf{LAD_2}$ is strongly normalizable if there is no infinite reduction sequence

$M \to M_1 \to M_2 \to \dots$

beginning with M.

$|M|$ is the maximum number of the reduction steps of the reduction sequences from M, which exists by König's lemma.

4 Strong normalization for terms of the system $\mathbf{LAD_2}$

Definition 4.1 SN denotes the set of strongly normalizable quasi-terms.

For $n \geq 0$, a quasi-term M is n-strongly normalizable if $MM_1 \dots M_n$ is strongly normalizable for all strongly normalizable quasi-terms $M_1, \dots, M_n$.

SN_n denotes the set of n-strongly normalizable terms.

Definition 4.2 (Saturated set) A set S of quasi-terms is saturated if the following hold.

(SN) $S \subset SN$,

$(VarConst)$ $\alpha\vec{R} \in S$ if $\vec{R} \in SN$ (α is u or the term constant c_2),

(β) $(\lambda\alpha.M)N\vec{R} \in S$ if $M[\alpha := N]\vec{R} \in S$ and $N \in SN$ (α is x, u or X),

$(pair1)$ $\quad \langle M,N\rangle p_0\vec{R}\in S$ if $M\vec{R}\in S$ and $N\in SN$,

$(pair2)$ $\quad \langle M,N\rangle p_1\vec{R}\in S$ if $N\vec{R}\in S$ and $M\in SN$,

$(\vee)$ $\quad (M,N,L)_{u,v}\vec{R}\in S$ if $N[u:=Mp_1]\vec{R}\in S$ and $L[v:=Mp_1]\vec{R}\in S$ and $M\in SN$,

$(\exists)$ $\quad (M,N)_{x,u}\vec{R}\in S$ if $N[x:=Mp_0,u:=Mp_1]\vec{R}\in S$ and $M\in SN$.

Definition 4.3 A set valuation is a function which maps every predicate variable to a saturated set.

For a set valuation σ, a predicate variable X, and a saturated set S, the set valuation $\sigma[X:=S]$ is defined by $(\sigma[X:=S])(X)=S$ and $(\sigma[X:=S])(Y)=\sigma(Y)$ for $X\not\equiv Y$.

For a type A and a set valuation σ, the set $\overline{A}\sigma$ of quasi-terms is defined by induction on A as follows.

$\overline{q\vec{t}}\sigma=SN$,

$\overline{X\vec{t}}\sigma=\sigma(X)$,

$M\in\overline{A\to B}\sigma$ iff $MN\in\overline{B}\sigma$ for every N in $\overline{A}\sigma$,

$M\in\overline{A\,\&\,B}\sigma$ iff $Mp_0\in\overline{A}\sigma$ and $Mp_1\in\overline{B}\sigma$,

$M\in\overline{\forall xA}\sigma$ iff $MN\in\overline{A}\sigma$ for every quasi-term $N\in SN$,

$\overline{F\vee G}\sigma=SN$,

$M\in\overline{\exists xA}\sigma$ iff $Mp_1\in\overline{A}\sigma$,

$M\in\overline{\bot}\sigma$ iff $Mp_\bot\in S$ for every saturated set S,

$M\in\overline{\forall X^nA}\sigma$ iff $MN\in\overline{A}(\sigma[X:=S])$ for every quasi-term $N\in SN_n$ and every saturated set S.

Remark. (1) The simple definition of $\overline{F\vee G}\sigma$ works because the formulas F and G are both atomic.

(2) Impredicativity is treated by the saturated sets. Second order variables range over the set of saturated sets in the definition of $\overline{A}\sigma$.

Notation. $|\vec{R}|=\Sigma_i|R_i|$ where $\vec{R}$ is the sequence $R_1\ldots R_n$. $lh(\vec{R})$ is the length of the sequence $\vec{R}$.

The next lemma proves that the set SN satisfies the conditions $(\vee)$ and $(\exists)$ of the definition of a saturated set.

Lemma 4.4 *(1) $(M,N,L)_{u,v}\vec{R}\in SN$ if $N[u:=Mp_1]\vec{R}$, $L[v:=Mp_1]\vec{R}$ and M are in SN.*

(2) $(M,N)_{x,u}\vec{R}\in SN$ if $N[x:=Mp_0,u:=Mp_1]\vec{R}$ and M are in SN.

Proof.

N,L are in SN in both (1) and (2) since $N[u:=Mp_1]\vec{R}$, $L[v:=Mp_1]\vec{R}$ of (1) and $N[x:=Mp_0,u:=Mp_1]\vec{R}$ of (2) are in SN.

We will prove (1) and (2) simultaneously by induction on a pair $(|M|+lh(\vec{R}),|N|+|L|+|\vec{R}|)$.

(1) Assume $N[u:=Mp_1]\vec{R}$ and $L[v:=Mp_1]\vec{R}$ and M are in SN. We will show that if $(M,N,L)_{u,v}\vec{R}\to K$, then $K\in SN$. Consider cases according to the reduction $\to$ used in $(M,N,L)_{u,v}\vec{R}\to K$.

Case $K \equiv (M', N, L)\vec{R}$ where $M \to M'$. We have $N[u := Mp_1] \to^* N[u := M'p_1] \in SN$ since $N[u := Mp_1] \in SN$. By IH for $|M'| < |M|$ with $lh(\vec{R})$ unchanged, $K \in SN$.

Case $K \equiv (M, N', L)\vec{R}$ where $N \to N'$. IH for $|N'| < |N|$ with $|M|$ and $lh(\vec{R})$ unchanged.

Case $K \equiv (M, N, L')\vec{R}$ where $L \to L'$. Similar to the previous case.

Case $K \equiv (M, N, L)\vec{R'}$ where $\vec{R} \to \vec{R'}$. IH for $|\vec{R'}| < |\vec{R}|$ with $|M|, |N|, |L|$ unchanged and $lh(\vec{R}) = lh(\vec{R'})$.

Case K is $N[u := Mp_1]\vec{R}$ or $L[v := Mp_1]\vec{R}$. By the assumption.

Case $(M, N, L)_{u,v}R\vec{R} \to (M, NR, LR)_{u,v}\vec{R} \equiv K$. We can assume $u, v \notin FV(R)$. By the assumption, $(NR)[u := Mp_1]\vec{R}, (LR)[v := Mp_1]\vec{R} \in SN$. By IH for $\vec{R}$ with $|M|$ unchanged and $lh(\vec{R}) + 1 = lh(R\vec{R})$, we have $(M, NR, LR)_{u,v}\vec{R} \in SN$.

Case $((P, Q)_{x,w}, N, L)_{u,v}\vec{R} \to (P, (Q, N, L)_{u,v})_{x,w}\vec{R} \equiv K$. We can assume $x, w \notin FV(N), FV(L)$. By the assumption and the reduction $(P, Q)_{x,w} \to Q[x := Pp_0, w := Pp_1]$, $N[u := Q[x := Pp_0, w := Pp_1]p_1]\vec{R}$, $L[v := Q[x := Pp_0, w := Pp_1]p_1]\vec{R}$, and $Q[x := Pp_0, w := Pp_1]$ are in SN. By IH for $|Q[x := Pp_0, w := Pp_1]| < |(P, Q)_{x,w}|$, $(Q[x := Pp_0, w := Pp_1], N, L)_{u,v}\vec{R}$ is in SN. Hence $(Q, N, L)_{u,v}[x := Pp_0, w := Pp_1]\vec{R}$ is in SN. By the assumption, P is in SN. By IH (2) for $|P| < |(P, Q)_{x,w}|$, $(P, (Q, N, L)_{u,v})_{x,w}\vec{R}$ is in SN. Note that we have $|P| < |(P, Q)_{x,w}|$ because $(P', Q)_{x,w}$ reduces to $Q[x := P'p_0, w := P'p_1]$ when P reduces to P'.

Case $((P, Q_1, Q_2)_{w_1,w_2}, N, L)_{u,v}\vec{R} \to$
$$(P, (Q_1, N, L)_{u,v}, (Q_2, N, L)_{u,v})_{w_1,w_2}\vec{R}.$$

We can assume $w_1, w_2 \notin FV(N), FV(L)$. By $(P, Q_1, Q_2)_{w_1,w_2} \to Q_1[w_1 := Pp_1]$ and the assumption, $N[u := Q_1[w_1 := Pp_1]p_1]\vec{R}$, $L[v := Q_1[w_1 := Pp_1]p_1]\vec{R}$, and $Q_1[w_1 := Pp_1]$ are in SN. By IH for $|Q_1[w_1 := Pp_1]| < |(P, Q_1, Q_2)_{w_1,w_2}|$, $(Q_1[w_1 := Pp_1], N, L)_{u,v}\vec{R}$ is in SN. Hence $(Q_1, N, L)_{u,v}[w_1 := Pp_1]\vec{R}$ is in SN. Similarly, $(Q_2, N, L)_{u,v}[w_2 := Pp_1]\vec{R}$ is in SN. From the assumption, P is in SN. By IH for $|P| < |(P, Q_1, Q_2)_{u,v}|$, $(P, (Q_1, N, L)_{u,v}, (Q_2, N, L)_{u,v})_{w_1,w_2}\vec{R}$ is in SN.

(2) Similar to (1). □

Lemma 4.5 *SN is a saturated set.*

Proof.

We check all conditions in the definition of the saturated sets.

(VarConst). Induction on $|\vec{R}|$.

(β). Assume $M[\alpha := N]\vec{R}$ and N are in SN. We will prove $(\lambda\alpha.M)N\vec{R} \in SN$ by induction on $|M| + |N| + |\vec{R}|$. Assume $(\lambda\alpha.M)N\vec{R} \to K$. Consider all cases according to the definition of the reduction $\to$ used in $(\lambda\alpha.M)N\vec{R} \to K$.

Case $K \equiv (\lambda\alpha.M')N\vec{R}$ where $M \to M'$. By IH for $|M'| < |M|$.

Case $K \equiv (\lambda\alpha.M)N'\vec{R}$ where $N \to N'$. By IH for $|N'| < |N|$.

Case $K \equiv (\lambda\alpha.M)N\vec{R'}$ where $\vec{R} \to \vec{R'}$. By IH for $|\vec{R'}| < |\vec{R}|$.

Case $K \equiv M[\alpha := N]\vec{R}$. By the assumption.

The conditions $(pair1)$ and $(pair2)$ are proved similarly to (β).

The conditions $(\vee)$ and $(\exists)$ are established in Lemma 4.4. □

Theorem 4.6 *The set $\overline{A}\sigma$ is saturated for every type A and every set valuation σ.*

Proof. Induction on A.

Case $q\vec{t}$. By Lemma 4.5.

Case $X\vec{t}$. $\overline{X\vec{t}}\sigma = \sigma(X)$ is saturated by the definition of σ.

Case $A \to B$.

(SN). Suppose $M \in \overline{A \to B}\sigma$. By IH A (VarConst), $w \in \overline{A}\sigma$. By the definition of $\overline{A \to B}\sigma$, $Mw \in \overline{B}\sigma$. By IH for B (SN), we have $Mw \in SN$. Hence M SN.

(VarConst). Let α and $\vec{R}$ be those in the condition (VarConst). Take arbitrary $M \in \overline{A}\sigma$. By IH for A (SN), we have $M \in SN$. By IH for B (VarConst), we have $\alpha\vec{R}M \in \overline{B}\sigma$. Hence $\alpha\vec{R} \in \overline{A \to B}\sigma$.

(β). Suppose $M[\alpha := N]\vec{R} \in \overline{A \to B}\sigma$, and $N \in SN$. Take arbitrary $P \in \overline{A}\sigma$. By the definition of $\overline{A \to B}\sigma$, we have $M[\alpha := N]\vec{R}P \in \overline{B}\sigma$. By IH for B (β), $(\lambda\alpha.M)N\vec{R}P \in \overline{B}\sigma$. Hence $(\lambda\alpha.M)N\vec{R} \in \overline{A \to B}\sigma$.

Other conditions are similar to (β).

Case $F \vee G$. By Lemma 4.5, since $\overline{F \vee G}\sigma = SN$.

The other cases are similar to Case $A \to B$. □

Definition 4.7 A valuation ρ is a function which maps every first order variable to an SN quasi-term, every term variable u to a term belonging to a saturated set, and every predicate variable of arity n to an SN_n quasi-term.

For a valuation ρ, a term variable u, and a term M belonging to a saturated set, the valuation $\rho[u := M]$ is defined by $(\rho[u := M])(u) = M$ and $(\rho[u := M])(\alpha) = \rho(\alpha)$ for a first order variable, a second order variable, or a term variable α such that $\alpha \not\equiv u$.

For a valuation ρ, a first order variable x, and an SN quasi-term M, the valuation $\rho[x := M]$ is defined similarly. For a valuation ρ, a predicate variable X^n, and an SN_n quasi-term M, the valuation $\rho[X := M]$ is defined similarly.

For a quasi-term M and a valuation ρ, $M\rho$ is defined as $M[\vec{x} := \rho(\vec{x}), \vec{u} := \rho(\vec{u}), \vec{X} := \rho(\vec{X})]$ where all the free variables of M are among $\vec{x}, \vec{u}, \vec{X}$.

The next lemma says that formulas and abstraction terms of $\mathbf{LAD_2}$ under substitution ρ are strongly normalizable.

Lemma 4.8 *Let ρ be a valuation.*

(1) $s\rho$ is in SN, if s is a first order term.

(2) $A\rho$ is in SN, if A is a formula.

(3) $\lambda\vec{x}.M$ is in SN_n, if $M[\vec{x} := \vec{N}]$ is in SN for all $\vec{N} \in SN$ where $lh(\vec{x}) = lh(\vec{N}) = n$.

(4) $T\rho$ is in SN_n if T is an abstraction term of arity n.

Proof. (1) Induction on s.

(2) Induction on A.

(3) We show that $\vec{N} \in SN$ and $M[\vec{x} := \vec{N}] \in SN$ imply $(\lambda\vec{x}.M)\vec{N} \in SN$ by induction on the pair $(lh(\vec{N}), |M| + |\vec{N}|)$.

Suppose $(\lambda\vec{x}.M)\vec{N} \to K$. We will show K is in SN. Consider cases according to the reduction $\to$ of $(\lambda\vec{x}.M)\vec{N} \to K$.

Case $(\lambda\vec{x}.M)\vec{N} \to (\lambda\vec{x}.M')\vec{N}$. By IH for $|M'| < |M|$.

Case $(\lambda\vec{x}.M)\vec{N} \to (\lambda\vec{x}.M)\vec{N}'$. By IH for $|N'| < |N|$.

Case $(\lambda x_1\vec{x}_2.M)N_1\vec{N}_2 \to (\lambda\vec{x}_2.M[x_1 := N_1])\vec{N}_2$ where $x_1\vec{x}_2 \equiv \vec{x}$ and $N_1\vec{N}_2 \equiv \vec{N}$. By the assumption, we have $M[x_1 := N_1][\vec{x}_2 := \vec{N}_2] \in SN$. By IH for $lh(\vec{N}_2) < lh(\vec{N})$, we have $(\lambda\vec{x}_2.M[x_1 := N_1])\vec{N}_2 \in SN$.

Therefore $(\lambda\vec{x}.M)\vec{N}$ is in SN.

(4) If $T \equiv X$, then the claim follows from $\rho(X) \in SN_n$. Suppose $T \equiv \lambda\vec{x}.A$ and $lh(\vec{x}) = n$. Assume $\vec{N}$ are in SN and $lh(\vec{N}) = n$. Let $\rho' = \rho[\vec{x} := \vec{N}]$. By (2), $A\rho'$ is in SN. $(A\rho)[\vec{x} := \vec{N}]$ is in SN. Hence $\vec{N} \in SN$ implies $(A\rho)[\vec{x} := \vec{N}] \in SN$. By (3), $\lambda\vec{x}.A\rho$ is in SN_n, that is, $T\rho$ is in SN_n. □

Lemma 4.9 *For any set valuation σ, the following hold.*

(1) $\overline{A[x := s]}\sigma = \overline{A}\sigma$.

(2) $\overline{A_X[T]}\sigma = \overline{A}(\sigma[X := \overline{(X\vec{0})_X[T]\sigma}])$ *where X is of arity n and $\vec{0}$ is the sequence of 0's of length n.*

(3) $\overline{A}(\sigma[X := S]) = \overline{A}\sigma$ *if $X \notin FV(A)$.*

Note. If $\{T\}$ is an abbreviation for $\overline{(X\vec{0})_X[T]\sigma}$, the claim (2) states that $\overline{A_X[T]}\sigma = \overline{A}(\sigma[X := \{T\}])$.

Proof. (1) and (3) are proved by induction on A.

(2) Induction on A.

Case $X\vec{t}$. If $T = Y$, then $LHS = \overline{Y\vec{t}}\sigma$ and by (1) it equals $\overline{Y\vec{0}}\sigma = RHS$. If $T = \lambda\vec{x}.A$, then $LHS = \overline{A[\vec{x} := \vec{t}]}\sigma$ and by (1) it equals $\overline{A[\vec{x} := \vec{0}]}\sigma = RHS$.

Other cases are straightforward. □

The next statement is central in this paper.

Theorem 4.10 (Reducibility) *Suppose ρ is a valuation, σ is a set valuation and $\rho(\vec{u}) \in \overline{\vec{B}}\sigma$.*

If $\vec{u} : \vec{B} \vdash M : A$, then $M\rho \in \overline{A}\sigma$.

Proof. We use induction on the proof of $\vec{u} : \vec{B} \vdash M : A$. Consider cases according to the last rule. sat (SN) denotes the condition (SN) of the definition of the saturated sets. sat (β) and so forth are similar.

Case $u : A$. $u\rho \equiv \rho(u) \in \overline{A}\sigma$ by the assumption.

Case

$$\frac{\begin{array}{c}[u : A] \\ \vdots \\ M : B\end{array}}{\lambda u.M : A \to B}$$

Assume $N \in \overline{A}\sigma$. Let $\rho' = \rho[u := N]$. By IH, $M\rho' \in \overline{B}\sigma$, that is, $(M\rho)[u := N] \in \overline{B}\sigma$. By sat (SN), N is in SN. By sat (β), $(\lambda u.M\rho)N \in \overline{B}\sigma$. By the definition of $\overline{A \to B}\sigma$, $\lambda u.M\rho \in \overline{A \to B}\sigma$. Hence $(\lambda u.M)\rho \in \overline{A \to B}\sigma$.

Case
$$\frac{M : A \to B \quad N : A}{MN : B}$$
By IH, $M\rho \in \overline{A \to B}\sigma$ and $N\rho \in \overline{A}\sigma$. Hence $(M\rho)(N\rho) \in \overline{B}\sigma$, that is, $(MN)\rho \in \overline{B}\sigma$.

Cases $(\&I)$ and $(\&E)$ are straightforward.

Case
$$\frac{M : A}{\lambda x.M : \forall x A}$$
Assume N is in SN. Let $\rho' = \rho[x := N]$. By IH, $M\rho' \in \overline{A}\sigma$, that is, $(M\rho)[x := N] \in \overline{A}\sigma$. By sat (β), $(\lambda x.M\rho)N \in \overline{A}\sigma$. Hence $(\lambda x.M)\rho \in \overline{\forall x A}\sigma$.

Case
$$\frac{M : \forall x A}{Ms : A[x := s]}$$
By IH, $M\rho \in \overline{\forall x A}\sigma$. By Lemma 4.8 (1), $s\rho$ is in SN. Hence $(M\rho)(s\rho) \in \overline{A}\sigma$. By Lemma 4.9 (1), $(Ms)\rho \in \overline{A[x := s]}\sigma$.

Case
$$\frac{M : A[x := s]}{\langle s, M\rangle : \exists x A}$$
By IH, $M\rho \in \overline{A[x := s]}\sigma$. By Lemma 4.9 (1), $M\rho \in \overline{A}\sigma$. By Lemma 4.8 (1), $s\rho$ is in SN. By sat $(pair2)$, $\langle s\rho, M\rho\rangle p_1 \in \overline{A}\sigma$. Hence $\langle s, M\rangle\rho \in \overline{\exists x A}\sigma$.

Case
$$\frac{M : \exists x A \quad \begin{array}{c}[u : A] \\ \vdots \\ N : C\end{array}}{(M, N)_{x,u} : C}$$
By IH, $M\rho \in \overline{\exists x A}\sigma$. By sat (SN), $M\rho$ is in SN. By the definition of $\overline{\exists x A}\sigma$, $M\rho p_1 \in \overline{A}\sigma$. Let $\rho' = \rho[x := M\rho p_0, u := M\rho p_1]$. By IH, $N\rho' \in \overline{C}\sigma$, that is, $(N\rho)[x := M\rho p_0, u := M\rho p_1] \in \overline{C}\sigma$. By sat $(\exists)$, $(M\rho, N\rho)_{x,u} \in \overline{C}\sigma$, that is, $(M, N)_{x,u}\rho \in \overline{C}\sigma$.

Case
$$\frac{M : F}{\langle 0, M\rangle : F \vee G}$$
By IH, $M\rho \in \overline{F}\sigma = SN$. Hence $\langle 0, M\rangle\rho \equiv \langle 0, M\rho\rangle \in SN = \overline{F \vee G}\sigma$.

Case
$$\frac{M : F \vee G \quad \begin{array}{c}[u : F] \\ \vdots \\ N : C\end{array} \quad \begin{array}{c}[v : G] \\ \vdots \\ L : C\end{array}}{(M, N, L)_{u,v} : C}$$
By IH, $M\rho$ is in SN. Hence $M\rho p_1$ is in SN. Let $\rho' = \rho[u := M\rho p_1]$. By IH, $N\rho' \in \overline{C}\sigma$, that is, $(N\rho)[u := M\rho p_1] \in \overline{C}\sigma$. Similarly $(L\rho)[v := M\rho p_1] \in \overline{C}\sigma$. By sat $(\vee)$, $(M\rho, N\rho, L\rho)_{u,v} \in \overline{C}\sigma$, that is, $(M, N, L)_{u,v}\rho \in \overline{C}\sigma$.

Case
$$\frac{M : A}{\lambda X.M : \forall X A}$$
Assume N is in SN_n and S is a saturated set. Let $\rho' = \rho[X := N]$ and $\sigma' = \sigma[X := S]$. By the variable condition, $\rho'(\vec{u}) \in \overline{\vec{B}}\sigma'$. By IH, $M\rho' \in \overline{A}\sigma'$.

Hence $(M\rho)[X := N] \in \overline{A}(\sigma[X := S])$. By sat (β), $(\lambda X.M\rho)N \in \overline{A}(\sigma[X := S])$. By the definition of $\overline{\forall X A}\sigma$, $(\lambda X.M)\rho \in \overline{\forall X A}\sigma$.

Case

$$\frac{M : \forall X A}{MT : A_X[T]}$$

By IH, $M\rho \in \overline{\forall X A}\sigma$. By Lemma 4.8 (4), $T\rho$ is in SN_n. By Theorem 4.6, $\overline{(X\vec{0})_X[T]\sigma}$ is a saturated set. By the definition of $\overline{\forall X A}\sigma$, $(M\rho)(T\rho) \in \overline{A}(\sigma[X := \overline{(X\vec{0})_X[T]\sigma}])$. By Lemma 4.9 (2), $(MT)\rho \in \overline{A_X[T]}\sigma$.

Case

$$\frac{M : \bot}{Mp_\bot : A}$$

By IH, $M\rho \in \overline{\bot}\sigma$. By the definition of $\overline{\bot}\sigma$, $M\rho p_\bot \equiv Mp_\bot\rho \in \overline{A}\sigma$.

Case $c_0 : q_0 0$. Obviously c_0 is in SN.

Case $c_2 : q_0 t \to q_1 t \to \bot$. Assume M, N are in SN and S is a saturated set. By sat (VarConst), $c_2 MNp_\bot \in S$. Hence $c_2 MN \in \overline{\bot}\sigma$. Hence $c_2\rho \equiv c_2 \in \overline{q_0 t \to q_1 t \to \bot}\sigma$. □

Theorem 4.11 (Strong Normalization) *If $M : A$, then M is strongly normalizable.*

Proof. Let $\rho = Id$ and $\sigma(X) = SN$. By the theorem 4.10, $M\rho \in \overline{A}\sigma$. By $M\rho \equiv M$ and sat (SN) for $\overline{A}\sigma$, we have $M \in SN$. □

Remark. (1) In this section, the definition of the reduction is used only in the proofs of Lemmas 4.4 and 4.5 and Lemma 4.8 (1),(2) and (3). Note that Lemma 4.8 uses the definition to show some claims like the statement that $\vec{M} \in SN$ implies $f\vec{M} \in SN$. The definition of the typing is used only in the proof of Theorem 4.10.

(2) If the system $\mathbf{LAD_2}$ had non-atomic disjunction, it would not be strongly normalizing. An example of non-termination for that system is the term M where

$N \equiv \lambda u.(\langle 0, u\rangle, v, wu)_{v,w}$,

$M \equiv NN$.

Then we have $\vdash M : X \to X$ in the extended $\mathbf{LAD_2}$ by using the non-atomic disjunctions $X \vee (X \to X)$ and $(X \to X) \vee ((X \to X) \to (X \to X))$. We have $\vdash N : X \to X$ by letting $u : X$, $v : X$, and $w : X \to X$ in N and using $X \vee (X \to X)$. Similarly we also have $\vdash N : (X \to X) \to (X \to X)$. Therefore $\vdash M : X \to X$ holds. On the other hand, N reduces $\lambda u.\langle 0, u\rangle p_1 u$ by $(\vee 2)$, and the latter reduces to $\lambda u.uu$. Hence M is not strongly normalizable, because $M \to^* (\lambda u.uu)(\lambda u.uu)$ holds.

5 Simulation of disjunction in $\mathbf{LAD_2}$

Suppose A and B are formulas. The system $\mathbf{LAD_2}$ simulates the formula $A \vee B$ by the formula

$\exists x((q_0 x \vee q_1 x) \,\&\, ((q_0 x \to A) \,\&\, (q_1 x \to B)))$

where $x \notin A, B$.

The inference

$$\frac{A}{A \vee B}\,(\vee I)$$

is simulated by the proof

$$\dfrac{\dfrac{\dfrac{q_0 0}{q_0 0 \vee q_1 0} \quad \dfrac{\dfrac{A}{q_0 0 \to A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{q_0 0 \to q_1 0 \to \bot \quad q_0 0}{q_1 0 \to \bot} \quad \overset{1}{q_1 0}}{\bot}}{B}}{q_1 0 \to B}\,1}{(q_0 0 \to A) \,\&\, (q_1 0 \to B)}}{(q_0 0 \vee q_1 0) \,\&\, ((q_0 0 \to A) \,\&\, (q_1 0 \to B))}}{\exists x((q_0 x \vee q_1 x) \,\&\, ((q_0 x \to A) \,\&\, (q_1 x \to B)))}$$

The inference

$$\dfrac{A \vee B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\,(\vee E)$$

is simulated by the proof

$$\dfrac{\exists x D \qquad \dfrac{\dfrac{\overset{2}{D}}{q_0 x \vee q_1 x} \quad \begin{matrix}\dfrac{\dfrac{\overset{2}{D}}{(q_0 x \to A) \,\&\, (q_1 x \to B)}}{q_0 x \to A} \quad \overset{1}{q_0 x} \\ \hline A \\ \vdots \\ C\end{matrix} \quad \begin{matrix}\dfrac{\dfrac{\overset{2}{D}}{(q_0 x \to A) \,\&\, (q_1 x \to B)}}{q_1 x \to B} \quad \overset{1}{q_1 x} \\ \hline B \\ \vdots \\ C\end{matrix}}{C}\,1}{C}\,2$$

where

$D \equiv (q_0 x \vee q_1 x) \,\&\, ((q_0 x \to A) \,\&\, (q_1 x \to B))$.

Let us define a translation from $NJ\forall^2$ into $\mathbf{LAD_2}$.

Definition 5.1 (Translation) For a formula A of $NJ\forall^2$, the formula $\tilde{A}$ of $\mathbf{LAD_2}$ is defined as follows.

$\tilde{A} \equiv A \quad (A \equiv q\vec{t}, X\vec{t}, \bot)$,

$\widetilde{A \to B} \equiv \tilde{A} \to \tilde{B}$,

$\widetilde{A \,\&\, B} \equiv \tilde{A} \,\&\, \tilde{B}$,

$\widetilde{\forall \alpha A} \equiv \forall \alpha \tilde{A} \quad (\alpha \equiv x, X)$,

$\widetilde{A \vee B} \equiv \exists x((q_0 x \vee q_1 x) \,\&\, ((q_0 x \to \tilde{A}) \,\&\, (q_1 x \to \tilde{B}))) \quad (x \text{ fresh})$,

$\widetilde{\exists x A} \equiv \exists x \tilde{A}$.

For an abstraction term T of $NJ\forall^2$, the abstraction term $\tilde{T}$ of $\mathbf{LAD_2}$ is defined by

$\tilde{X} \equiv X$,

$\widetilde{\lambda \vec{x}.A} \equiv \lambda \vec{x}.\tilde{A}$.

For a term M of $NJ\forall^2$, the quasi-term $\tilde{M}$ of $\mathbf{LAD_2}$ is defined as follows.

$\tilde{u^A} \equiv u$,

$\widetilde{\lambda u^A.M} \equiv \lambda u.\tilde{M}$,

$\widetilde{MN} \equiv \tilde{M}\tilde{N}$,
$\widetilde{\langle M, N\rangle} \equiv \langle \tilde{M}, \tilde{N}\rangle$,
$\widetilde{Mp_i} \equiv \tilde{M}p_i \quad (i \equiv 0, 1)$,
$\widetilde{\lambda x.M} \equiv \lambda x.\tilde{M}$,
$\widetilde{Ms} \equiv \tilde{M}s$,
$\widetilde{\langle 0, M\rangle^A} \equiv \langle 0, \langle\langle 0, c_0\rangle, \langle \lambda v_f.\tilde{M}, \lambda v_1.c_2c_0v_1p_\perp\rangle\rangle\rangle \quad (v_f \text{ fresh})$,
$\widetilde{\langle 1, M\rangle^A} \equiv \langle 1, \langle\langle 1, c_1\rangle, \langle \lambda v_0.c_2v_0c_1p_\perp, \lambda v_f.\tilde{M}\rangle\rangle\rangle \quad (v_f \text{ fresh})$,
$\widetilde{(M, N, L)}_{u^A, v^B} \equiv$
$(\tilde{M}, (wp_0, \tilde{N}[u := wp_1p_0u_0], \tilde{L}[v := wp_1p_1u_1])_{u_0,u_1})_{x,w}$
$(u_0, u_1, w \text{ fresh})$,
$\widetilde{\langle s, M\rangle^A} \equiv \langle s, \tilde{M}\rangle$,
$\widetilde{(M, N)}_{x,u^A} \equiv (\tilde{M}, \tilde{N})_{x,u}$,
$\widetilde{Mp_A} \equiv \tilde{M}p_\perp$,
$\widetilde{\lambda X.M} \equiv \lambda X.\tilde{M}$,
$\widetilde{MT} \equiv \tilde{M}\tilde{T}$.

We sometimes write $(M)^\sim$ instead of $\tilde{M}$.

Lemma 5.2 *(1)* $\widetilde{A[x := s]} \equiv \tilde{A}[x := s]$.
(2) $\widetilde{A_X[T]} \equiv \tilde{A}_X[\tilde{T}]$.
(3) $(M[x := s, u^A := N, X := T])^\sim \equiv \tilde{M}[x := s, u := \tilde{N}, X := \tilde{T}]$ *holds. This claim holds also when variable capturing is allowed in the substitution.*
(4) In $\mathbf{LAD_2}$, *if* $\Gamma, u : B \vdash M : A$ *and* $\Gamma \vdash N : B$, *then* $\Gamma \vdash M[u := N] : A$.
(5) If $x, u^A, X \notin FV(M)$, *then* $x, u, X \notin FV(\tilde{M})$.
(6) If M *has just one free occurrence of the variable* u^A, *then* $\tilde{M}$ *has just one free occurrence of the variable* u.

Proof. (1)(2) Induction on A.
(3)(5)(6) Induction on M.
(4) Induction on the proof of $\Gamma, u : B \vdash M : A$. □

Theorem 5.3 (Soundness of the translation) *(1) If* $u^{\vec{B}} : \vec{B} \vdash M : A$ *in* $NJ\forall^2$, *then* $\vec{u} : \vec{\tilde{B}} \vdash \tilde{M} : \tilde{A}$ *in* $\mathbf{LAD_2}$.
(2) If $M \to N$ *in* $NJ\forall^2$, *then* $\tilde{M} \to^+ \tilde{N}$ *in* $\mathbf{LAD_2}$.

Proof. (1) Induction on the proof of $u^{\vec{B}} : \vec{B} \vdash M : A$.

Case $(\to I)$.

$$\dfrac{\begin{array}{c}[u^A : A] \\ \vdots \\ M : B\end{array}}{\lambda u^A.M : A \to B}$$

By IH, $\tilde{\Gamma}, u : \tilde{A} \vdash \tilde{M} : \tilde{B}$. By $(\to I)$, $\tilde{\Gamma} \vdash \lambda u.\tilde{M} : \tilde{A} \to \tilde{B}$, and this is $\tilde{\Gamma} \vdash \widetilde{\lambda u^A.M} : \widetilde{A \to B}$.

Case $(\vee I1)$.

$$\dfrac{M : A}{\langle 0, M\rangle : A \vee B}$$

We have $c_0 : q_0 0$ and $\langle 0, c_0\rangle : q_0 0 \vee q_1 0$.

By IH, $\tilde{M} : \tilde{A}$. Hence $\lambda v_f.\tilde{M} : q_0 0 \to \tilde{A}$.

We also have $c_2 : q_0 0 \to q_1 0 \to \bot$; $c_0 : q_0 0$; $c_2 c_0 : q_1 0 \to \bot$. Assume $v_1 : q_1 0$. We get $c_2 c_0 v_1 : \bot$; $c_2 c_0 v_1 p_\bot : \tilde{B}$. Hence $\lambda v_1.c_2 c_0 v_1 p_\bot : q_1 0 \to \tilde{B}$.

By combining them, we get $\langle\langle 0, c_0\rangle, \langle \lambda v_f.\tilde{M}, \lambda v_1.c_2 c_0 v_1 p_\bot\rangle\rangle : (q_0 0 \vee q_1 0) \,\&\, ((q_0 0 \to \tilde{A}) \,\&\, (q_1 0 \to \tilde{B}))$ and $\langle 0, \langle\langle 0, c_0\rangle, \langle \lambda v_f.\tilde{M}, \lambda v_1.c_2 c_0 v_1 p_\bot\rangle\rangle\rangle : \exists x((q_0 x \vee q_1 x) \,\&\, ((q_0 x \to \tilde{A}) \,\&\, (q_1 x \to \tilde{B})))$. The latter is $\widetilde{\langle 0, M\rangle} : \widetilde{A \vee B}$.

Case $(\vee I2)$ is similar to Case $(\vee I1)$.

Case $(\vee E)$.

$$\dfrac{M : A \vee B \qquad \begin{matrix}[u^A : A] \\ \vdots \\ N : C\end{matrix} \qquad \begin{matrix}[v^B : B] \\ \vdots \\ L : C\end{matrix}}{(M, N, L)_{u^A, v^B} : C}$$

Let $D \equiv (q_0 x \vee q_1 x) \,\&\, ((q_0 x \to \tilde{A}) \,\&\, (q_1 x \to \tilde{B}))$.

Assume $w : D$.

We have $wp_0 : q_0 x \vee q_1 x$.

We have $wp_1 p_0 : q_0 x \to A$. Assume $u_0 : q_0 x$. Then $wp_1 p_0 u_0 : A$. By IH, $\tilde{\Gamma}, u : \tilde{A} \vdash \tilde{N} : \tilde{C}$. By Lemma 5.2 (4), $\tilde{\Gamma}, u_0 : q_0 x, w : D \vdash \tilde{N}[u := wp_1 p_0 u_0] : \tilde{C}$.

We also have $wp_1 p_1 : q_1 x \to B$. Assume $u_1 : q_1 x$. Then $wp_1 p_1 u_1 : B$. By IH, $\tilde{\Gamma}, v : \tilde{B} \vdash \tilde{L} : \tilde{C}$. By Lemma 5.2 (4), $\tilde{\Gamma}, u_1 : q_1 x, w : D \vdash \tilde{L}[v := wp_1 p_1 u_1] : \tilde{C}$.

By combining wp_0 and the two terms for $\tilde{C}$, we obtain $\tilde{\Gamma}, w : D \vdash (wp_0, \tilde{N}[u := wp_1 p_0 u_0], \tilde{L}[v := wp_1 p_1 u_1])_{u_0, u_1} : \tilde{C}$. By IH, $\tilde{M} : \widetilde{A \vee B}$, that is, $\tilde{M} : \exists x D$. Hence $\tilde{\Gamma} \vdash (\tilde{M}, (wp_0, \tilde{N}[u := wp_1 p_0 u_0], \tilde{L}[v := wp_1 p_1 u_1])_{u_0, u_1})_{x, w} : \tilde{C}$, that is, $\tilde{\Gamma} \vdash \widetilde{(M, N, L)}_{u^A, v^B} : \tilde{C}$.

Other cases are similar to $(\to I)$.

(2) We prove this claim by induction on the definition of the reduction $M \to N$. Consider the same cases as in the definition of the reductions in $NJ\forall^2$.

Case $(\to)$ $(\lambda u^A.M)N \to M[u^A := N]$. We have $\widetilde{(\lambda u^A.M)N} \equiv (\lambda u.\tilde{M})\tilde{N} \to \tilde{M}[u := \tilde{N}]$. By Lemma 5.2 (3), the right hand side is $\widetilde{M[u^A := N]}$.

Case $(\vee 1)$ $(\langle 0, M\rangle, N, L)_{u^A, v^B} \to N[u^A := M]$. We have

$$\begin{aligned}
& ((\langle 0, M\rangle, N, L)_{u^A, v^B})^\sim \\
\equiv\ & (\langle 0, \langle\langle 0, c_0\rangle, \langle \lambda v_f.\tilde{M}, \lambda v_1.c_2 c_0 v_1 p_\bot\rangle\rangle\rangle, \\
& \quad (wp_0, \tilde{N}[u := wp_1 p_0 u_0], \tilde{L}[v := wp_1 p_1 u_1])_{u_0, u_1})_{x, w} \\
\to\ & ((\langle 0, \langle\langle 0, c_0\rangle, \langle \lambda v_f.\tilde{M}, \lambda v_1.c_2 c_0 v_1 p_\bot\rangle\rangle\rangle) p_1 p_0, \\
& \tilde{N}[u := (\langle 0, \langle\langle 0, c_0\rangle, \langle \lambda v_f.\tilde{M}, \lambda v_1.c_2 c_0 v_1 p_\bot\rangle\rangle\rangle) p_1 p_1 p_0 u_0], \\
& \tilde{L}[v := (\langle 0, \langle\langle 0, c_0\rangle, \langle \lambda v_f.\tilde{M}, \lambda v_1.c_2 c_0 v_1 p_\bot\rangle\rangle\rangle) p_1 p_1 p_1 u_1])_{u_0, u_1} \\
\to^+ & (\langle 0, c_0\rangle, \tilde{N}[u := (\lambda v_f.\tilde{M}) u_0], \tilde{L}[v := (\lambda v_1.c_2 c_0 v_1 p_\bot) u_1])_{u_0, u_1} \\
\to\ & \tilde{N}[u := (\lambda v_f.\tilde{M}) u_0][u_0 := \langle 0, c_0\rangle p_1] \\
\to^*\ & \tilde{N}[u := \tilde{M}]
\end{aligned}$$

By Lemma 5.2 (3), it is $\widetilde{N[u^A := M]}$.

Case ($\vee 2$) is similar to Case ($\vee 1$).

Case ($\exists$) $(\langle s, M\rangle, N)_{x,u^A} \to N[x := s, u^A := M]$. We have $\widetilde{(\langle s, M\rangle, N)_{x,u^A}} \equiv (\langle s, \tilde{M}\rangle, \tilde{N})_{x,u} \to \tilde{N}[x := \langle s, \tilde{M}\rangle p_0, u := \langle s, \tilde{M}\rangle p_1] \to^* \tilde{N}[x := s, u := \tilde{M}]$. By Lemma 5.2 (3), the right hand side is $\widetilde{N[x := s, u^A := M]}$.

Case ($perm\vee$) $(M, N, L)_{u^A,v^B} R \to (M, NR, LR)_{u^A,v^B}$. We can assume $u^A, v^B \notin R$. By Lemma 5.2 (5), $u, v \notin \tilde{R}$. We have

$$
\begin{aligned}
& ((M, N, L)_{u^A,v^B} R)^\sim \\
\equiv\ & (\tilde{M}, (wp_0, \tilde{N}[u := wp_1p_0u_0], \tilde{L}[v := wp_1p_1u_1])_{u_0,u_1})_{x,w}\tilde{R} \\
\to\ & (\tilde{M}, (wp_0, \tilde{N}[u := wp_1p_0u_0], \tilde{L}[v := wp_1p_1u_1])_{u_0,u_1}\tilde{R})_{x,w} \\
\to\ & (\tilde{M}, (wp_0, \tilde{N}[u := wp_1p_0u_0]\tilde{R}, \tilde{L}[v := wp_1p_1u_1]\tilde{R})_{u_0,u_1})_{x,w} \\
\equiv\ & (\tilde{M}, (wp_0, (\tilde{N}\tilde{R})[u := wp_1p_0u_0], (\tilde{L}\tilde{R})[v := wp_1p_1u_1])_{u_0,u_1})_{x,w} \\
\equiv\ & ((M, NR, LR)_{u^A,v^B})^\sim.
\end{aligned}
$$

Cases ($perm \vee \vee$) and ($perm \vee \exists$) are similar to ($perm\vee$).

Case (Congr) $M \to M'$ where $N \to N'$ holds and M' is obtained from M by replacing just one occurrence of N by N'. Suppose $N : A$. There exists a term L and a term variable u^A such that M is obtained from L by replacing just one occurrence of u^A by N by allowing variable capturing. Then M' is obtained from L by replacing u^A by N' in the same way.

By Lemma 5.2 (6), $\tilde{L}$ has just one occurrence of the variable u. By Lemma 5.2 (3), $\tilde{M}$ is obtained from $\tilde{L}$ by replacing u by $\tilde{N}$ by allowing variable capturing. Similarly $\tilde{M}'$ is obtained from $\tilde{L}$ by replacing u by $\tilde{N}'$ by allowing variable capturing.

By IH, we have $\tilde{N} \to^+ \tilde{N}'$. Hence $\tilde{M} \to^+ \tilde{M}'$ holds.

Other cases are similar to Case ($\to$). □

Theorem 5.4 (Strong normalization) *If $M : A$ in $NJ\forall^2$, then M is strongly normalizable in $NJ\forall^2$.*

Proof. Assume $M \to M_1 \to M_2 \to \cdots$. By Theorem 5.3 (2), we have the sequence $\tilde{M} \to^+ \tilde{M}_1 \to^+ \cdots$ in $\mathbf{LAD_2}$. On the other hand, by Theorem 5.3 (1), we have $\tilde{M} : \tilde{A}$ in $\mathbf{LAD_2}$. By Theorem 4.11, we have $\tilde{M} \in SN$ in $\mathbf{LAD_2}$ and this leads to a contradiction. □

6 Concluding Remarks

As a referee pointed out, an attempt to extend our proof to second order existential quantification is blocked because a union of saturated sets is not in general a saturated set. Our proof has worked for second order universal quantification, since an arbitrary intersection of saturated sets is a saturated set.

A union of saturated sets fails to be saturated because of the condition ($\vee$) in the definition of saturated sets. Given two saturated sets S_1 and S_2, to show that $S_1 \cup S_2$ is saturated, we need to check the condition $(\vee)$. We assume

$N[u := Mp_1]\vec{R} \in S_1 \cup S_2$,

$L[v := Mp_1]\vec{R} \in S_1 \cup S_2$.

The assumption of the condition may be the case that

$N[u := Mp_1]\vec{R} \in S_1$,

$L[v := Mp_1]\vec{R} \in S_2$.

We do not see any way to derive the conclusion of the condition

$(M, N, L)_{u,v}\vec{R} \in S_1 \cup S_2$.

Our proof can be extended to second-order existential quantification with one of the following restrictions:

- Atomic second-order existential quantification: we impose the condition that T is atomic for
$$\frac{M : A_X[T]}{\langle T, M\rangle^{\exists XA} : \exists XA}\ (\exists^2 I)$$
Then we can define $M \in \overline{\exists X^n A}\sigma$ as $Mp_0 \in SN_n$ and $Mp_1 \in \overline{A}(\sigma[X := SN])$, since X ranges over only atomic abstraction terms and we can choose SN as the saturated set for the value of X.
- Positive second-order existential quantification: we have the restriction that X occurs only positively in A for $(\exists^2 I)$. Then $\overline{A}(\sigma[X := S])$ is increasing with respect to S. We can define $M \in \overline{\exists X^n A}\sigma$ as $Mp_0 \in SN_n$ and $Mp_1 \in \overline{A}(\sigma[X := S])$ for some saturated set S. Then we can choose $\overline{A}(\sigma[X := SN])$ as the covering of $\overline{A}(\sigma[X := S_1])$ and $\overline{A}(\sigma[X := S_2])$ in the proof of Theorem 4.6, the case $\exists XA$, the condition $(\vee)$.
- Negative second-order existential quantification: we have the restriction that X occurs only negatively in A for $(\exists^2 I)$. Then $\overline{A}(\sigma[X := S])$ is decreasing with respect to S. We can define $M \in \overline{\exists X^n A}\sigma$ as $Mp_0 \in SN_n$ and $Mp_1 \in \overline{A}(\sigma[X := S])$ for some saturated set S. Then we can choose $\overline{A}(\sigma[X := S_1 \cap S_2])$ as the covering of $\overline{A}(\sigma[X := S_1])$ and $\overline{A}(\sigma[X := S_2])$ in the proof of Theorem 4.6, the case $\exists XA$, the condition $(\vee)$.
- Disjunction is prohibited. Because of absence of the condition $(\vee)$, we do not need use a union of saturated sets. So we can define $M \in \overline{\exists X^n A}\sigma$ as $Mp_0 \in SN_n$ and $Mp_1 \in \overline{A}(\sigma[X := S])$ for some saturated set S.

We can prove strong normalization of those systems in the same way as this paper. However, our proof does not work for the second-order existential quantification without any restrictions of them.

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References

[1] H.P. Barendregt, Lambda calculi with types, In: *Hand book of Logic in Computer Science, Vol. 2,* S. Abramsky et al, eds. (Oxford University Press, 1992) 118–309.

[2] R. David and K. Nour, A short proof of the strong normalization of classical natural deduction with disjunction, *Journal of Symbolic Logic* **68** (4) (2003) 1277–1288.

[3] P. de Groote, On the Strong Normalisation of Intuitionistic Natural Deduction with Permutative-Conversions, *Information and Computation* **178** (2002) 441–464.

[4] J.H. Gallier, On Girard's "candidats de reducibilite", In: *Logic and Computer Science,* P. Odifreddi, ed. (Academic Press, 1990) 123–203.

[5] J.Y. Girard, Une extension de l'interprétation de Gödel à l'analyse, et son application à l'élimination des coupures dans l'analyse et la théorie des types, In: *Proceedings of the Second Scandinavian Logic Symposium,* J.E. Fenstad, ed. (North-Holland, 1971) 63–92.

[6] J.Y. Girard, Interprétation fonctionelle et élimination des coupures de l'arithmétique d'ordre supérieur, Thèse de Doctorat d'Etat, Université de Paris VII (1972).

[7] J.Y. Girard, P. Taylor and Y. Lafont, *Proofs And Types* (Cambridge University Press, 1989).

[8] W.A. Howard, The formulae-as-types notion of construction, In: *To H.B. Curry: essays on combinatory logic, lambda calculus and formalism,* J.P. Seldin and J.R. Hindley, eds. (Academic Press, 1980) 480–490.

[9] F. Joachimski and R. Matthes, Short proofs of normalization for the simply-typed lambda-calculus, permutative conversions and Gödel's T, *Archive for Mathematical Logic* **42** (2003) 59–87.

[10] D. Leivant, Strong normalization for arithmetic, *Lecture Notes in Mathematics* **500** (1975) 182–197.

[11] P. Martin-Löf, Hauptsatz for the theory of species, In: *Proceedings of the Second Scandinavian Logic Symposium,* J.E. Fenstad, ed. (North-Holland, 1971) 217–233.

[12] R. Matthes, Non-Strictly Positive Fixed-Points for Classical Natural Deduction, *Annals of Pure and Applied Logic,* to appear.

[13] G. Mints, Existential instantiation and strong normalization, *Lecture Notes in Computer Science* **1234** (1997) 258–265.

[14] D. Prawitz, *Natural Deduction* (Almquist and Wiksell, 1965).

[15] D. Prawitz, Ideas and results of proof theory, In: *Proceedings of the Second Scandinavian Logic Symposium,* J.E. Fenstad, ed. (North-Holland, 1971) 235–307.

[16] W.W. Tait, A realizability interpretation of the theory of species, *Lecture Notes in Mathematics* **453** (1975) 240–251.

[17] W.W. Tait, The completeness of Heyting first-order logic, *Journal of Symbolic Logic* **68** (3) (2003) 751–763.

[18] A.S. Troelstra, Metamathematical Investigation of Intuitionistic Arithmetic and Analysis, *Lecture Notes in Mathematics* **344** (1973).

[19] A.S. Troelstra and H. Schwichtenberg, *Basic Proof Theory, Second Edition* (Cambridge Univ Press, 2000).