Nonlinear Laplacian for Digraphs and its Applications to Network Analysis

Yuichi Yoshida
(National Institute of Informatics)

Question: Can we develop spectral graph theory for digraphs?

- Spectral graph theory analyzes graph properties via eigenpairs of associated matrices (in particular, Laplacian).
- Well established for undirected graphs.
- Extensions for digraphs are largely unexplored although many real-world networks are directed.

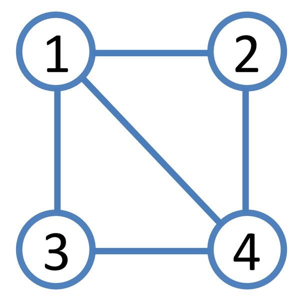
Definition

The Laplacian for an undirected graph G

Adjacency matrix A_G
Degree Matrix D_G

Laplacian $L_G := D_G - A_G$
Normalized Laplacian:
$\mathcal{L}_G := D_G^{-1/2} L_G D_G^{-1/2}$

G



$$L_G = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

The Laplacian for a digraph G (proposed)

Laplacian $L_G: \mathbb{R}^V \to \mathbb{R}^V$ transforms $\mathbf{x} \in \mathbb{R}^V$ as follows:
- Construct an undirected graph H as follows:
- For each arc $u \to v$
  - If $\mathbf{x}(u) \geq \mathbf{x}(v)$, add an undirected edge $\{u, v\}$.
  - Otherwise, add self-loops to u and v.
- Output $L_H\mathbf{x}$.

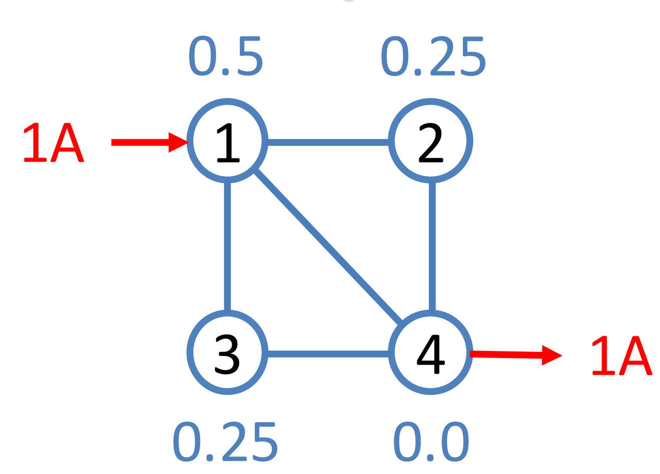
Normalized Laplacian: $\mathcal{L}_G : x \mapsto D_G^{-1/2} L_G (D_G^{-1/2}\mathbf{x})$

Interpretation via electrical circuits

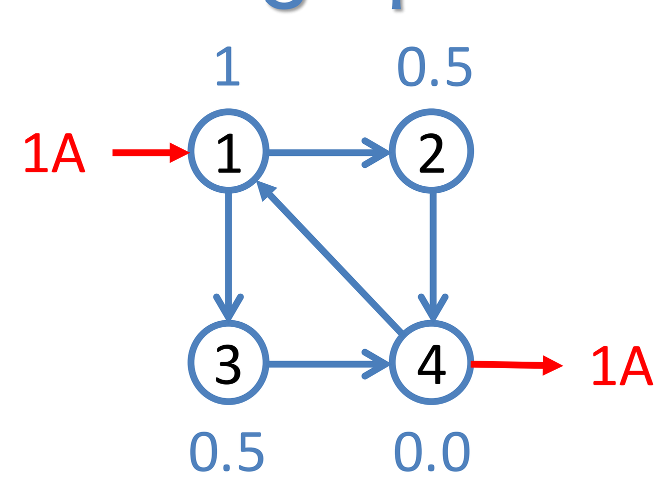
Regard G as an electrical circuit.
- Graph: edge = resistance of 1Ω
- Digraph: arc = diode of 1Ω (current flows only one way)

For each $u \in V$, flow a current of $\mathbf{b}(u)$ amperes to u.
The voltages $\mathbf{x} \in \mathbb{R}^V$ of vertices is given by $L_G(\mathbf{x}) = \mathbf{b}$.

Graph



Digraph



Properties

$(\lambda, \mathbf{v})$ is an eigenpair of $\mathcal{L}_G$ if $\mathcal{L}_G(\mathbf{v}) = \lambda \mathbf{v}$
Trivial eigenpair $(\lambda_1, \mathbf{v}_1)$ with $\lambda_1 = 0$.

For any subspace U of positive dimension, $\Pi_U \mathcal{L}_G$ has an eigenpair. (Π_U = Projection matrix to U)

⇒ Another eigenvalue of $\mathcal{L}_G$ exists by choosing $U = \mathbf{v}_1^{\perp}$.
Let λ_2 = the second smallest eigenvalue.

Algorithm

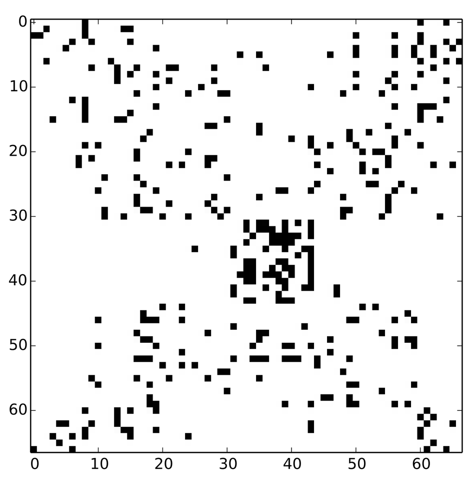
Computing λ_2 is (likely to be) NP-hard.

Suppose we start the diffusion process
$$d\boldsymbol{x} = -\Pi_U \mathcal{L}_G(\boldsymbol{x})dt$$
from a vector in the subspace $U = \mathbf{v}_1^{\perp}$.
- $\mathbf{x}$ converges to an eigenvector orthogonal to $\mathbf{v}_1$.
- Rayleigh quotient never increases during the process

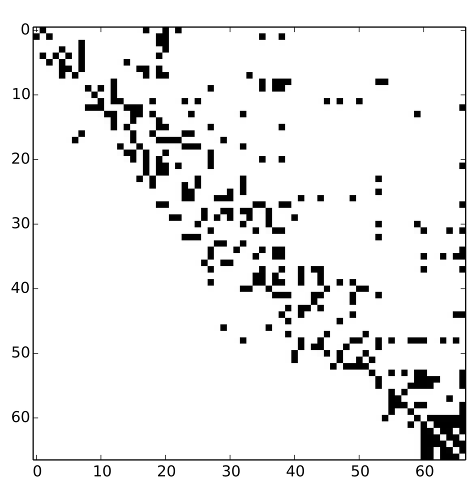
Visualization

Friendship network at a high school in Illinois
(u→v: u regards v as a friend)
Reorder vertices according to the eigenvector computed by the diffusion process

Chung's Laplacian



Proposed Laplacian

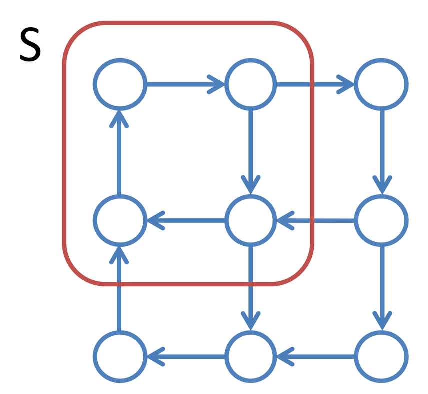


λ_2 is the minimum of
$$\sum_{u \to v} (\boldsymbol{x}(u) - \boldsymbol{x}(v))^2 \, \pi_u / d_u^+$$
subject to $x \neq 0$, $x \perp \mathbf{v}_1$.

$$\sum_{u \to v} (\max(\boldsymbol{x}(u) - \boldsymbol{x}(v), 0))^2$$

Community Detection

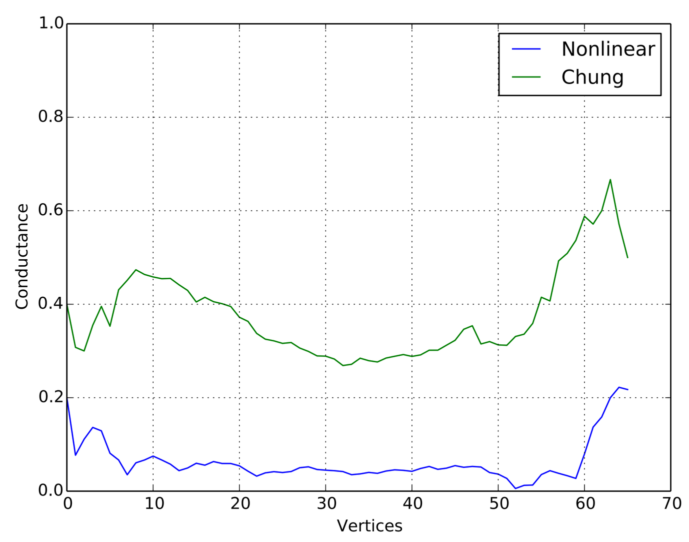
(Directed) conductance $\phi^+(S)$ of S:
$$\frac{\mathrm{cut}^+(S)}{\min(\mathrm{vol}(S), \mathrm{vol}(V - S))}$$
vol(S): total degree of $u \in S$.
$\mathrm{cut}^+(S)$: # of arcs from S to V-S



$\phi^+(S) = 2/12 = 1/6$

Cheeger's inequality for digraphs:
$$\lambda_2 \leq \min_S \phi^+(S) \leq 2\sqrt{\lambda_2}$$

Conductance of the set of the first k vertices after reordering.



Email: yyoshida@nii.ac.jp