Matroid representation of clique complexes*

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Abstract

In this paper, we approach the quality of a greedy algorithm for the maximum weighted clique problem from the viewpoint of matroid theory. More precisely, we consider the clique complex of a graph (the collection of all cliques of the graph) which is also called a flag complex, and investigate the minimum number k such that the clique complex of a given graph can be represented as the intersection of k matroids. This number k can be regarded as a measure of "how complex a graph is with respect to the maximum weighted clique problem" since a greedy algorithm is a k-approximation algorithm for this problem. For any $k > 0$, we characterize graphs whose clique complexes can be represented as the intersection of k matroids. As a consequence, we can see that the class of clique complexes is the same as the class of the intersections of partition matroids. Moreover, we determine how many matroids are necessary and sufficient for the representation of all graphs with n vertices. This number turns out to be $n-1$. Other related investigations are also given.

Keywords: Abstract simplicial complex, Clique complex, Flag complex, Independence system, Matroid intersection, Partition matroid

1 Introduction

An independence system is a family of subsets of a nonempty finite set such that all subsets of a member of the family are also members of the family. A lot of combinatorial optimization problems can be seen as optimization problems on the corresponding independence systems. For example, in the minimum cost spanning tree problem, we want to find a maximal set with minimum total weight in the collection of all forests of a given graph, which is an independence system. Other problems like the maximum weighted matching problem and the maximum weighted clique problem are also such problems. More examples are provided by Korte & Vygen [18]. In this paper, we study independence systems arising from the maximum weighted clique problem.

A clique in a graph is a subset of the vertex set which induces a complete graph. In the maximum weighted clique problem, we are given a graph and a weight function on the vertex set, and we want to find a clique which maximizes the total weight of its vertices. As is well known, the maximum weighted clique problem is NP-hard even if the weight function is constant [11]. This means that there exists no polynomial-time algorithm for this problem unless P = NP. Moreover, Håstad [13] proved that there exists no polynomial-time algorithm for this problem which approximates the optimal value within a factor $n^{1-\epsilon}$ for any $\epsilon > 0$ unless NP = ZPP. (Here, n stands for the number of vertices in a given graph.) Therefore, the maximum clique problem is deeply inapproximable. Thus, one wants to determine classes of graphs for which we can perform well. To do that, we adapt the viewpoint from independence systems and matroids. For the maximum weighted clique problem, we consider the family of all cliques of a graph as an independence system. Such an independence system is called a clique complex.

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It is known that every independence system can be represented as the intersection of a finite number of matroids. Jenkyns [14] and Korte & Hausmann [17] showed that, for the maximum weighted base problem on an independence system which can be represented as the intersection of k matroids, a natural greedy algorithm approximates the optimal value within a factor k. (Their result can be seen as a generalization of the validity of the greedy algorithm for matroids, shown by Rado [24] and Edmonds [7], although their results showed that the validity of the greedy algorithm even characterizes matroids.) Thus, this number k is a measure of "how complex an independence system is with respect to the corresponding optimization problem."

Here, we want to state the importance of clique complexes in fields other than combinatorial optimization. In extremal combinatorics, the f-vector of a clique complex (namely, the sequence $(f_{-1}, f_0, f_1, \dots, f_{n-1})$ where f_{i-1} is the number of cliques of size i in a graph) is studied in connection with Turán's problem. (See Bollobás [2].) Related to that, in algebraic combinatorics, problems on the roots of the f-polynomial of a clique complex are studied. For example, Hamidoune [12] asked whether the f-polynomial of the clique complex of a graph whose complement is claw-free has only real roots.[1] Also, Charney & Davis [4] made a conjecture on a clique complex which triangulates a homology sphere of odd dimension. For this topic, see Stanley's survey article [25]. Finally, in topological combinatorics, when we refer to the topology of a graph, sometimes it means the topology of the clique complex of the graph. The topology of clique complexes plays an important role especially when one investigates Hall-type theorems in hypergraphs [1, 19, 20]. Similarly, when we refer to the topology of a partially ordered set, it usually means the topology of the order complex of the partially ordered set, which turns out to be a clique complex.

In this paper, we investigate how many matroids we need for the representation of the clique complex of a graph as their intersection. We show that the clique complex of a given graph G is the intersection of k matroids if and only if there exists a family of k stable-set partitions of G such that every edge of $\overline{G}$ (the complement of G) is contained in a stable set of some stable-set partition in the family. This theorem implies that the problem of determining whether or not the clique complex of a given graph has a representation by k matroids belongs to NP (for any $k > 0$). This is not a trivial fact since in general the size of an independence system can be exponential. As another consequence, we show that the class of clique complexes is the same as the class of the intersections of partition matroids. This may give a new direction of research to attack some open problems on clique complexes.

Formerly, Fekete, Firla & Spille [9] investigated the same problem for matching complexes, and they characterized a graph whose matching complex is the intersection of k matroids, for every natural number k. Since the matching complexes form a subclass of the class of clique complexes, we can observe that some of their results can be derived from our theorems as corollaries.

With our main theorem, we deduce more results. First of all, we consider an extremal problem related to our theorem. Namely, we determine how many matroids are necessary and sufficient for the representation of all graphs with n vertices. This number turns out to be $n-1$. Secondly, we investigate the case of two matroids more thoroughly. This case is especially important since the maximum weighted base problem can be solved exactly in polynomial time for the intersection of two matroids [10]. (Namely, in this case, the maximum weighted clique problem can be solved in polynomial time for any non-negative weight vector by Frank's algorithm [10].) There, we find out that the algorithm by Protti & Szwarcfiter [23] checks whether a given clique complex has a representation by two matroids or not in polynomial time. Additionally, we show that the clique complex of a graph G is the intersection of k matroids if and only if G itself is the intersection of k matroids. (Here, we regard graphs themselves as independence systems of rank 2.) Thus, this reveals the intimate relationship between a graph and its clique complex in terms of matroid intersection.

The organization of this paper is as follows. In Section 2, we introduce a terminology on independence systems. The proof of the main theorem is given in Section 3. Some of the immediate consequences of the main theorem are also given there. In Section 4, we consider an extremal problem related to our theorem. In Section 5, we investigate the case of two matroids. In Section 6, we study a graph itself as an independence system and relate it to our theorem. In Section 7, we deduce some results by Fekete, Firla & Spille [9] from our theorems. We conclude with Section 8.

[1] Recently, this conjecture has been settled affirmatively by Chudnovsky & Seymour [5].

2 Preliminaries

2.1 Graphs

We assume the basic concepts in graph theory (see, e.g., Diestel's book [6]). Here, we fix our notations. In this paper, all graphs are finite and simple unless stated otherwise. For a graph $G = (V, E)$ we denote the subgraph induced by $V' \subseteq V$ by $G[V']$. The complement of G is denoted by $\overline{G}$. The vertex set and the edge set of a graph $G = (V, E)$ are denoted by $V(G)$ and $E(G)$, respectively. A complete graph and a cycle with n vertices are denoted by K_n and C_n, respectively. The maximum degree, the chromatic number and the edge-chromatic number (or the chromatic index) of a graph G are denoted by $\Delta(G)$, $\chi(G)$ and $\chi'(G)$, respectively. A *clique* of a graph $G = (V, E)$ is a subset $C \subseteq V$ such that the induced subgraph $G[C]$ is complete. A *stable set* of a graph $G = (V, E)$ is a subset $S \subseteq V$ such that the induced subgraph $G[S]$ has no edge.

2.2 Independence systems and matroids

Now we introduce the notions of independence systems and matroids. For details of them, see Oxley's book [22]. Given a non-empty finite set V, an *independence system* on V is a non-empty family $\mathcal{I}$ of subsets of V satisfying: $X \in \mathcal{I}$ implies $Y \in \mathcal{I}$ for all $Y \subseteq X \subseteq V$. The set V is called the *ground set* of this independence system. In the literature, an independence system is also called an *abstract simplicial complex*. A *matroid* is an independence system $\mathcal{I}$ additionally satisfying the following *augmentation axiom*: for $X, Y \in \mathcal{I}$ with $|X| > |Y|$ there exists $z \in X \setminus Y$ such that $Y \cup \{z\} \in \mathcal{I}$. For an independence system $\mathcal{I}$, a set X is called *independent* if $X \in \mathcal{I}$, and X is called *dependent* otherwise. A *base* of an independence system is a maximal independent set, and a *circuit* of an independence system is a minimal dependent set. (Notice that, in this paper, we use the word "circuit" only for independence systems, not for graphs. A circuit of a graph in a usual sense is referred to as a "cycle.") We denote the family of bases of an independence system $\mathcal{I}$ and the family of circuits of $\mathcal{I}$ by $\mathcal{B}(\mathcal{I})$ and $\mathcal{C}(\mathcal{I})$, respectively. Note that we can reconstruct an independence system $\mathcal{I}$ from $\mathcal{B}(\mathcal{I})$ or $\mathcal{C}(\mathcal{I})$ as $\mathcal{I} = \{X \subseteq V \mid X \subseteq B \text{ for some } B \in \mathcal{B}(\mathcal{I})\}$ and $\mathcal{I} = \{X \subseteq V \mid C \not\subseteq X \text{ for all } C \in \mathcal{C}(\mathcal{I})\}$. In particular, $\mathcal{B}(\mathcal{I}_1) = \mathcal{B}(\mathcal{I}_2)$ if and only if $\mathcal{I}_1 = \mathcal{I}_2$; similarly $\mathcal{C}(\mathcal{I}_1) = \mathcal{C}(\mathcal{I}_2)$ if and only if $\mathcal{I}_1 = \mathcal{I}_2$. We can see that all the bases of a matroid have the same size from the augmentation axiom, but it is not the case for an independence system in general.

Let $\mathcal{I}$ be a matroid on V. An element $x \in V$ is called a *loop* of $\mathcal{I}$ if $\{x\}$ is a circuit of $\mathcal{I}$. We say that $x, y \in V$ are *parallel* in $\mathcal{I}$ if $\{x, y\}$ is a circuit of the matroid $\mathcal{I}$. The next fact is well known.

Lemma 2.1 (see [22]). *For a matroid without a loop, the relation that "x is parallel to y" is an equivalence relation.*

Proof. Let $\mathcal{I}$ be a matroid on V without loop. Furthermore, let x and y be parallel in $\mathcal{I}$, and y and z be also parallel in $\mathcal{I}$. Then we claim that x and z are parallel in $\mathcal{I}$ as well (namely $\{x, z\}$ is a circuit of $\mathcal{I}$).

Suppose that $\{x, z\} \in \mathcal{I}$. Since $\mathcal{I}$ has no loop, it holds that $\{y\} \in \mathcal{I}$. By the augmentation axiom for matroids, we have that $\{x, y\} \in \mathcal{I}$ or $\{y, z\} \in \mathcal{I}$. However, this contradicts the assumption that x and y are parallel (implying $\{x, y\} \notin \mathcal{I}$) and y and z are parallel (implying $\{y, z\} \notin \mathcal{I}$). Therefore, it follows that $\{x, z\} \notin \mathcal{I}$. Since $\mathcal{I}$ has no loop, it holds that $\{x\} \in \mathcal{I}$ and $\{z\} \in \mathcal{I}$. This means that $\{x, z\}$ is a minimal dependent set (namely a circuit) of $\mathcal{I}$. □

Let $\mathcal{I}_1, \mathcal{I}_2$ be independence systems on the same ground set V. The *intersection* of $\mathcal{I}_1$ and $\mathcal{I}_2$ is just $\mathcal{I}_1 \cap \mathcal{I}_2$. The intersection of more independence systems is defined in a similar way. Note that the intersection of independence systems is also an independence system. In addition, note that the family of circuits of $\mathcal{I}_1 \cap \mathcal{I}_2$ is the family of the minimal sets in $\mathcal{C}(\mathcal{I}_1) \cup \mathcal{C}(\mathcal{I}_2)$, i.e.,

$$\mathcal{C}(\mathcal{I}_1 \cap \mathcal{I}_2) = \mathrm{MIN}(\mathcal{C}(\mathcal{I}_1) \cup \mathcal{C}(\mathcal{I}_2)).$$

(Here, the notation $\mathrm{MIN}(\mathcal{F})$ means that

$$\mathrm{MIN}(\mathcal{F}) := \{X \in \mathcal{F} \mid Y \not\subseteq X \text{ for every } Y \in \mathcal{F} \setminus \{X\}\}$$

for a set system $\mathcal{F}$.) The following well-known observation is crucial in this paper.

Lemma 2.2 (see [8, 9, 18]). *Every independence system can be represented as the intersection of a finite number of matroids on the same ground set.*

Proof. Denote the circuits of an independence system $\mathcal{I}$ by $C^{(1)}, \dots, C^{(m)}$ (i.e., $\mathcal{C}(\mathcal{I}) = \{C^{(1)}, \dots, C^{(m)}\}$), and consider the independence system $\mathcal{I}_i$ with a unique circuit $\mathcal{C}(\mathcal{I}_i) = \{C^{(i)}\}$ for each $i \in \{1, \dots, m\}$. Note that $\mathcal{I}_i$ is a matroid for each $i \in \{1, \dots, m\}$. Then, the family of the circuits of the intersection $\bigcap_{i=1}^m \mathcal{I}_i$ is nothing but $\{C^{(1)}, \dots, C^{(m)}\}$. Namely, $\mathcal{C}(\bigcap_{i=1}^m \mathcal{I}_i) = \{C^{(1)}, \dots, C^{(m)}\}$. Thus, we obtain that $\mathcal{C}(\mathcal{I}) = \mathcal{C}(\bigcap_{i=1}^m \mathcal{I}_i)$. Since the family of circuits determines an independence system uniquely, it follows that $\mathcal{I} = \bigcap_{i=1}^m \mathcal{I}_i$. □

Note that the matroids $\mathcal{I}_1, \dots, \mathcal{I}_m$ in the proof are actually graphic matroids. (A graphic matroid is an independence system isomorphic to the family of forests in a multigraph.) Therefore, Lemma 2.2 itself can be strengthened as "every independence system can be represented as the intersection of a finite number of graphic matroids on the same ground set," although it is not important for the discussion in the rest of the paper.

Due to Lemma 2.2, we are interested in the representation of an independence system as their intersection of matroids. From the construction in the proof of Lemma 2.2, we can see that the number of matroids which we need to represent an independence system $\mathcal{I}$ by the intersection is at most $|\mathcal{C}(\mathcal{I})|$. However, we might do better. In this paper, we investigate such a number for a clique complex.

3 Clique complexes and the main theorem

A graph gives rise to various independence systems. Among them, we study clique complexes.

The *clique complex* of a graph $G = (V, E)$ is the collection of all cliques of G. We denote the clique complex of G by $\mathfrak{C}(G)$. Note that the empty set is a clique and $\{v\}$ is also a clique for each $v \in V$. So we can see that the clique complex is actually an independence system on V. We also say that an independence system is a clique complex if it is isomorphic to the clique complex of some graph. Notice that a clique complex is also called a *flag complex* in the literature.

Here, we give some subclasses of the clique complexes. (We omit standard definitions.) (1) The family of the stable sets of a graph G is nothing but the clique complex of $\overline{G}$. (2) The family of the matchings of a graph G is the clique complex of the complement of the line graph of G, which is called the *matching complex* of G. (3) The family of the chains of a partially ordered set P is the clique complex of the comparability graph of P, which is called the *order complex* of P. (4) The family of the antichains of a partially ordered set P is the clique complex of the cocomparability graph (i.e., the complement of the comparability graph) of P.

The next lemma may be a folklore.

Lemma 3.1. *Let $\mathcal{I}$ be an independence system on a finite set V. Then, $\mathcal{I}$ is a clique complex if and only if the size of every circuit in $\mathcal{I}$ is two. In particular, the circuits of the clique complex of G are the edges of $\overline{G}$ (i.e., $\mathcal{C}(\mathfrak{C}(G)) = E(\overline{G})$).*

Proof. Let $\mathcal{I}$ be the clique complex of $G = (V, E)$. Since a single vertex $v \in V$ forms a clique, the size of each circuit in $\mathcal{I}$ is greater than one. Each dependent set of size two in $\mathcal{I}$ is an edge of the complement of G. Observe that they are minimal dependent sets since the size of each dependent set in $\mathcal{I}$ is greater than one. In order to show that they are the only minimal dependent sets, suppose that there exists a circuit C of size more than two in $\mathcal{I}$, for the contradiction. Then each two elements in C form an edge of G because of the minimality of C. Hence C is a clique in G. However, this is a contradiction to the assumption that C is dependent in $\mathcal{I}$ (i.e., not a clique in G).

Conversely, let $\mathcal{I}$ be an independence system on V and assume that the size of every circuit of $\mathcal{I}$ is two. Now construct a graph $G' = (V, E')$ with $E' = \{\{u, v\} \in \binom{V}{2} \mid \{u, v\} \notin \mathcal{C}(\mathcal{I})\}$, and consider the clique complex $\mathfrak{C}(G')$. By the opposite direction which we have just shown, we can see that a circuit of $\mathfrak{C}(G')$ is an edge of $\overline{G'}$, which is a circuit of $\mathcal{I}$. On the other hand, a circuit of $\mathcal{I}$, which is of size two, is an edge of $\overline{G'}$. Therefore we have that $\mathcal{C}(\mathfrak{C}(G')) = \mathcal{C}(\mathcal{I})$. This concludes that $\mathcal{I}$ is the clique complex of G'. □

Now, we start studying the number of matroids which we need for the representation of a clique complex as their intersection. For a graph G, denote by $\mu(G)$ the minimum number of matroids such that the clique complex $\mathfrak{C}(G)$ is

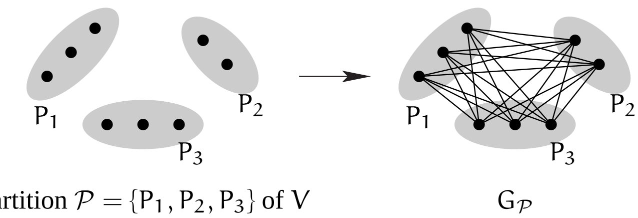


Figure 1: The correspondence of a partition matroid and a complete multipartite graph.

the intersection of them. Namely,

$$\mu(G) := \min\left\{ k \mid \mathfrak{C}(G) = \bigcap_{i=1}^{k} \mathcal{I}_i \text{ where } \mathcal{I}_1, \dots, \mathcal{I}_k \text{ are matroids} \right\}.$$

First, we characterize the graphs G satisfying $\mu(G) = 1$ (namely the graphs whose clique complexes are indeed matroids). To do this, we define a partition matroid. A *partition matroid* is a matroid $\mathcal{I}(\mathcal{P})$ associated with a partition $\mathcal{P} = \{P_1, P_2, \dots, P_r\}$ of V (that is, $V = \bigcup_{i=1}^{r} P_i$ and $P_i \cap P_j = \emptyset$ for all $i \neq j$), which is defined as

$$\mathcal{I}(\mathcal{P}) := \{I \subseteq V \mid |I \cap P_i| \leq 1 \text{ for all } i \in \{1, \dots, r\}\}.$$

Observe that $\mathcal{I}(\mathcal{P})$ is indeed a matroid. Being an independence system is clear. For the augmentation axiom, choose arbitrary two sets $X, Y \in \mathcal{I}(\mathcal{P})$ such that $|X| > |Y|$. Then, there must exist an index $i \in \{1, \dots, r\}$ such that $X \cap P_i \neq \emptyset$ and $Y \cap P_i = \emptyset$. Therefore, for a unique element $z \in X \cap P_i$, it holds that $Y \cup \{z\} \in \mathcal{I}(\mathcal{P})$.

Furthermore, observe that $\mathcal{I}(\mathcal{P})$ is a clique complex. Indeed we can see that $\mathcal{I}(\mathcal{P}) = \mathfrak{C}(G_\mathcal{P})$ as soon as we construct the following graph $G_\mathcal{P} = (V, E)$ from $\mathcal{P}$: two vertices $u, v \in V$ are adjacent in $G_\mathcal{P}$ if and only if u, v are elements of distinct partition classes in $\mathcal{P}$. See Figure 1 for an illustration.

An alternative argument is to observe that

$$\mathcal{C}(\mathcal{I}(\mathcal{P})) = \{\{u, v\} \in \binom{V}{2} \mid \{u, v\} \subseteq P_i \text{ for some } i \in \{1, \dots, r\}\}.$$

Then, we can find out that $\mathcal{I}(\mathcal{P})$ satisfies the condition in Lemma 3.1, which shows $\mathcal{I}(\mathcal{P})$ is a clique complex. Note that $G_\mathcal{P}$ constructed above is a complete r-partite graph with the partition $\mathcal{P}$. (In Figure 1, $G_\mathcal{P}$ is a complete tri-partite graph.) Particularly, this means that, if G is a complete multipartite graph, then $\mu(G) = 1$. In the following characterization of a matroidal clique complex, we claim that the converse also holds.

Lemma 3.2. *Let* G *be a graph. Then the following are equivalent.*

(1) *The clique complex of* G *is a matroid.*

(2) *The clique complex of* G *is a partition matroid.*

(3) G *is complete* r*-partite for some* r*.*

Note that the equivalence of (1) and (3) in the lemma is also noticed by Okamoto [21].

Proof. "(2) ⇒ (1)" is clear, and "(3) ⇒ (2)" is immediate from the discussion above. So we only have to show "(1) ⇒ (3)." Assume that the clique complex $\mathfrak{C}(G)$ is a matroid. By Lemma 3.1, every circuit of $\mathfrak{C}(G)$ is of size two, which corresponds to an edge of $\overline{G}$. Therefore, the elements of each circuit are parallel in $\mathfrak{C}(G)$. Since for every vertex $v \in V(G)$ we have $\{v\} \in \mathfrak{C}(G)$, we can see that $\mathfrak{C}(G)$ has no loop. Therefore, by Lemma 2.1, the parallel elements induce an equivalence relation on $V(G)$, which yields a partition $\mathcal{P} = \{P_1, \dots, P_r\}$ of $V(G)$ for some r. This equivalence relation can be said as "x and y are equivalent if and only if there is no edge between x and y in G." Thus, we can see that G is a complete r-partite graph with the vertex partition $\mathcal{P}$. □

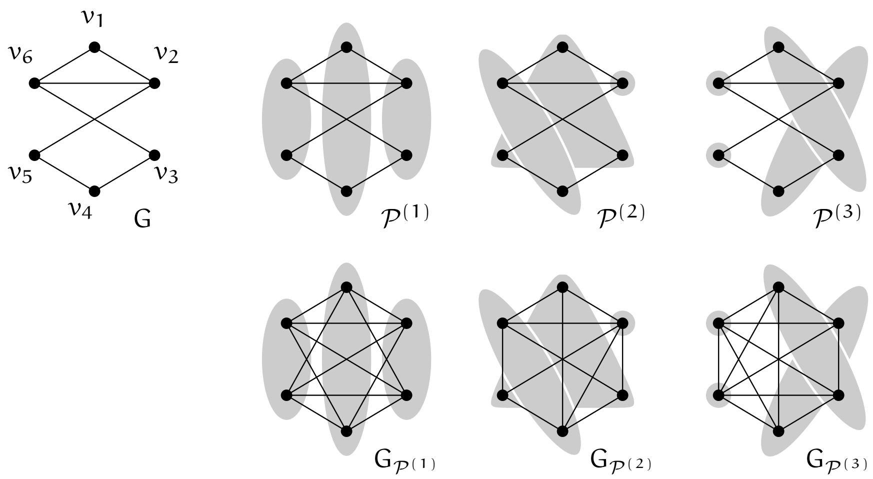


Figure 2: An example for Theorem 3.3.

For the case of more matroids, we use a stable-set partition. A *stable-set partition* of a graph $G = (V, E)$ is a partition $\mathcal{P} = \{P_1, \dots, P_r\}$ of V such that each P_i is a stable set of G. (Note that a stable-set partition is nothing else a proper coloring of a graph. However, here we are not interested in how many colors we need (i.e., the size of $\mathcal{P}$) as we do not study the proper coloring problem here.) The following theorem is the main result of this paper. It tells us how many matroids we need for the representation of a given clique complex.

Theorem 3.3. *Let* $G = (V, E)$ *be a graph. Then, the following are equivalent.*

(1) *The clique complex* $\mathfrak{C}(G)$ *can be represented as the intersection of* k *matroids (i.e.,* $\mu(G) \leq k$*).*

(2) *There exist* k *stable-set partitions* $\mathcal{P}^{(1)}, \dots, \mathcal{P}^{(k)}$ *of* G *which fulfill the following condition.*

Condition P*:*
$\{u, v\} \in \binom{V}{2}$ is an edge of $\overline{G}$ if and only if $\{u, v\} \subseteq S$ for some $S \in \bigcup_{i=1}^{k} \mathcal{P}^{(i)}$.

In particular, when Condition P *is fulfilled, it holds that* $\mathfrak{C}(G) = \bigcap_{i=1}^{k} \mathcal{I}(\mathcal{P}^{(i)})$.

Before proving Theorem 3.3, we illustrate the theorem by a pictorial example. Look at Figure 2. In the graph $G = (\{v_1, \dots, v_6\}, E)$, there are seven edges, and

$$\begin{aligned} \mathcal{P}^{(1)} &= \{\{v_1, v_4\}, \{v_2, v_3\}, \{v_5, v_6\}\}, \\ \mathcal{P}^{(2)} &= \{\{v_1, v_3, v_5\}, \{v_2\}, \{v_4, v_6\}\}, \\ \mathcal{P}^{(3)} &= \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5\}, \{v_6\}\} \end{aligned}$$

are stable-set partitions of G. We can see that these stable-set partitions meet Condition P, that is, for each $\{u, v\} \in E(\overline{G})$, it holds that $\{u, v\} \subseteq S$ for some $S \in \mathcal{P}^{(1)} \cup \mathcal{P}^{(2)} \cup \mathcal{P}^{(3)}$. For example, look at $\{v_1, v_5\} \in E(\overline{G})$. Then we have $\{v_1, v_3, v_5\} \in \mathcal{P}^{(2)}$ such that $\{v_1, v_5\} \subseteq \{v_1, v_3, v_5\}$. Indeed, the clique complex $\mathfrak{C}(G)$ can be written as the intersection $\mathcal{I}(\mathcal{P}^{(1)}) \cap \mathcal{I}(\mathcal{P}^{(2)}) \cap \mathcal{I}(\mathcal{P}^{(3)})$ of three partition matroids, or in other words, the intersection $\mathfrak{C}(G_{\mathcal{P}^{(1)}}) \cap \mathfrak{C}(G_{\mathcal{P}^{(2)}}) \cap \mathfrak{C}(G_{\mathcal{P}^{(3)}})$ of the clique complexes of complete multipartite graphs, which are partition matroids (Lemma 3.2).

The intuition behind Condition P in Theorem 3.3 is as follows. When we consider the clique complex $\mathfrak{C}(G)$ of a given graph G, we want to gather some complete multipartite graphs $G_1, \dots, G_k$ so that we can ensure that $\mathfrak{C}(G) = \bigcap_{i=1}^{k} \mathfrak{C}(G_i)$. Then an edge of $\overline{G}$ should not be an edge of G_i for all $i \in \{1, \dots, k\}$, and actually Condition P in Theorem 3.3 makes it sure that this requirement is satisfied.

To prove Theorem 3.3, we use the following lemmas.

Lemma 3.4. *Let* $G = (V, E)$ *be a graph. If the clique complex* $\mathfrak{C}(G)$ *can be represented as the intersection of* k *matroids (i.e.,* $\mu(G) \leq k$*), then there exist* k *stable-set partitions* $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(k)}$ *such that* $\mathfrak{C}(G) = \bigcap_{i=1}^{k} \mathcal{I}(\mathcal{P}^{(i)})$.

Proof. Assume that $\mathfrak{C}(G)$ is represented as the intersection of k matroids $\mathcal{I}_1, \ldots, \mathcal{I}_k$. Choose $\mathcal{I}_i$ arbitrarily ($i \in \{1, \ldots, k\}$). Then observe that there is no loop in $\mathcal{I}_i$. (Otherwise $\bigcap \mathcal{I}_i$ cannot be a clique complex.) Therefore, by Lemma 2.1, the parallel elements of $\mathcal{I}_i$ induce an equivalence relation on V. Let $\mathcal{P}^{(i)}$ be the partition of V arising from this equivalence relation. Then, we can see that the two-element circuits of $\mathcal{I}_i$ are the circuits of the partition matroid $\mathcal{I}(\mathcal{P}^{(i)})$, (i.e., $\{C \in \mathcal{C}(\mathcal{I}_i) \mid |C| = 2\} = \mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)}))$). Furthermore, by Lemma 3.1, it holds that

$$\begin{aligned} \mathcal{C}(\mathfrak{C}(G)) &= \mathrm{MIN}\left(\bigcup_{i=1}^{k} \mathcal{C}(\mathcal{I}_i)\right) = \mathrm{MIN}\left(\bigcup_{i=1}^{k} \{C \in \mathcal{C}(\mathcal{I}_i) \mid |C| = 2\}\right) \\ &= \bigcup_{i=1}^{k} \{C \in \mathcal{C}(\mathcal{I}_i) \mid |C| = 2\} = \bigcup_{i=1}^{k} \mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)})). \end{aligned}$$

Thus, we have obtained that $\mathcal{C}(\mathfrak{C}(G)) = \bigcup_{i=1}^{k} \mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)}))$. This concludes that $\mathfrak{C}(G) = \bigcap_{i=1}^{k} \mathcal{I}(\mathcal{P}^{(i)})$. □

Here is another lemma.

Lemma 3.5. *Let* $G = (V, E)$ *be a graph and* $\mathcal{P}$ *be a partition of* V. *Then* $\mathfrak{C}(G) \subseteq \mathcal{I}(\mathcal{P})$ *if and only if* $\mathcal{P}$ *is a stable-set partition of* G.

Proof. Assume that $\mathcal{P}$ is a stable-set partition of G. Choose $I \in \mathfrak{C}(G)$ arbitrarily. Then we have that $|I \cap P| \leq 1$ for each $P \in \mathcal{P}$ by the definitions of a clique and a stable set. Hence it follows that $I \in \mathcal{I}(\mathcal{P})$. Thus we have that $\mathfrak{C}(G) \subseteq \mathcal{I}(\mathcal{P})$.

Conversely, assume that $\mathfrak{C}(G) \subseteq \mathcal{I}(\mathcal{P})$ for a partition P of $V(G)$. Choose $P \in \mathcal{P}$ and a clique $K \in \mathfrak{C}(G)$ of G arbitrarily. From our assumption, we have that $K \in \mathcal{I}(\mathcal{P})$. Therefore, it holds that $|K \cap P| \leq 1$ from the definition of a partition matroid. This means that P is a stable set of G. Hence, $\mathcal{P}$ is a stable-set partition of G. □

Now it is time to prove Theorem 3.3.

Proof of Theorem 3.3. Assume that the clique complex $\mathfrak{C}(G)$ of a given graph $G = (V, E)$ is represented as the intersection of k matroids $\mathcal{I}_1, \ldots, \mathcal{I}_k$. From Lemma 3.4, $\mathfrak{C}(G)$ can be represented as the intersection of k matroids associated with some stable-set partitions $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(k)}$ of G. We show that these partitions $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(k)}$ fulfill Condition P. By Lemma 3.1, $\{u, v\}$ is an edge of $\overline{G}$ if and only if $\{u, v\}$ is a circuit of the clique complex $\mathfrak{C}(G)$. Then, we have that

$$\{u, v\} \in \mathcal{C}(\mathfrak{C}(G)) = \mathrm{MIN}\left(\bigcup_{i=1}^{k} \mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)}))\right) = \bigcup_{i=1}^{k} \mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)})).$$

(The last identity relies on the fact that the size of each circuit of a partition matroid is two.) This means that there exists at least one $i \in \{1, \ldots, k\}$ such that $\{u, v\} \in \mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)}))$. Since $\mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)})) = \{\{u, v\} \in \binom{V}{2} \mid \{u, v\} \subseteq S \text{ for some } S \in \mathcal{P}^{(i)}\}$, we can see that $\{u, v\} \subseteq S$ for some $S \in \mathcal{P}^{(i)}$ if and only if $\{u, v\}$ is an edge of $\overline{G}$.

Conversely, assume that we are given k stable-set partitions $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(k)}$ of V satisfying Condition P. We show that $\mathfrak{C}(G) = \bigcap_{i=1}^{k} \mathcal{I}(\mathcal{P}^{(i)})$. By Lemma 3.5, we can see that $\mathfrak{C}(G) \subseteq \mathcal{I}(\mathcal{P}^{(i)})$ for each $i \in \{1, \ldots, k\}$. This shows that $\mathfrak{C}(G) \subseteq \bigcap_{i=1}^{k} \mathcal{I}(\mathcal{P}^{(i)})$. In order to show that $\mathfrak{C}(G) \supseteq \bigcap_{i=1}^{k} \mathcal{I}(\mathcal{P}^{(i)})$, we only have to show that $\mathcal{C}(\mathfrak{C}(G)) \subseteq \bigcup_{i=1}^{k} \mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)}))$. Pick $C \in \mathcal{C}(\mathfrak{C}(G))$ arbitrarily. By Lemma 3.1 we can see that C is an edge of $\overline{G}$. Set $C = \{u, v\} \in E(\overline{G})$. From Condition P, there exists some $S \in \bigcup_{i=1}^{k} \mathcal{P}^{(i)}$ such that $\{u, v\} \subseteq S$. This means that $\{u, v\} \in \bigcup_{i=1}^{k} \mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)}))$. Thus we complete the proof. □

Now, let us look at some consequences of the discussion in this section. First of all, Theorem 3.3 implies that the clique complex $\mathfrak{C}(G)$ of a graph G can be represented as the intersection of k matroids if and only if $\mathfrak{C}(G)$ can be represented as the intersection of k *partition matroids arising from stable-set partitions of* G. Therefore, in order to find $\mu(G)$, it is sufficient to consider partition matroids arising from stable-set partitions of G. This considerably reduces the time/cost of the search.

In Lemma 3.4, we showed that, for a given graph G on the vertex set V whose clique complex $\mathfrak{C}(G)$ is the intersection of k matroids, we can find k partition matroids whose intersection is $\mathfrak{C}(G)$. Moreover, we can show the following "converse" statement.

Corollary 3.6. *For any collection of k partitions $\mathcal{P}^{(1)}, \mathcal{P}^{(2)}, \ldots, \mathcal{P}^{(k)}$ of a finite set V, there exists a graph G on V such that $\mathfrak{C}(G)$ is the intersection of the partition matroids $\mathcal{I}(\mathcal{P}^{(1)}), \mathcal{I}(\mathcal{P}^{(2)}), \ldots, \mathcal{I}(\mathcal{P}^{(k)})$.*

Proof. From a given collection of partitions $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(k)}$ of V, we construct a graph G as follows. The vertex set of G is V. Two vertices u and v are connected by an edge in G if and only if they do not lie in a common class of $\mathcal{P}^{(i)}$ for any $i \in \{1, \ldots, k\}$ (i.e., there exists no $S \in \mathcal{P}^{(i)}$ such that $\{u, v\} \subseteq S$ for any $i \in \{1, \ldots, k\}$). Then we can see that $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(k)}$ are stable-set partitions of G. Moreover, they satisfy Condition P in the statement of Theorem 3.3. Therefore, by Theorem 3.3, we can conclude that $\mathfrak{C}(G) = \bigcap_{i=1}^{k} \mathcal{I}(\mathcal{P}^{(i)})$. □

This leads to the following important consequence.

Corollary 3.7. *For every $k > 0$, the class of clique complexes which are the intersections of k matroids is the same as the class of the intersections of k partition matroids; in particular, the class of clique complexes is the same as the class of the intersections of partition matroids.*

Proof. Combine Lemma 3.4 and Corollary 3.6. □

At the end of this section, we want to note that Theorem 3.3 implies that the following decision problem belongs to NP.

Problem:	CLIQUE COMPLEX k-MATROID REPRESENTATION
Instance:	A graph G and a positive integer k
Question:	Is $\mu(G) \leq k$?

Let us state this fact as a corollary.

Corollary 3.8. CLIQUE COMPLEX k-MATROID REPRESENTATION *belongs to* NP.

Proof. This is not trivial since a matroid itself can have an exponential number of independent sets. However, from the viewpoint of Theorem 3.3, k stable-set partitions satisfying Condition P can be a certificate for the positive answer to the problem above. Since the size of stable-set partition is a polynomial of the size of a graph G and k is at most the number of vertices in G, these k stable-set partitions constitute a polynomial-size certificate. Furthermore, Condition P can be checked in polynomial time for a given graph and given k stable-set partitions of the graph. That is why the decision problem CLIQUE COMPLEX k-MATROID REPRESENTATION belongs to NP. □

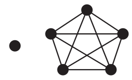
However, we do not know that CLIQUE COMPLEX k-MATROID REPRESENTATION belongs to P, or even to coNP. Possibly it could be NP-complete. When k is fixed, the status is somehow changed. For $k = 1$, due to Lemma 3.2 the problem can be solved in polynomial time because it is easy to check whether a graph is complete multipartite. The case of $k = 2$ is discussed in Section 5, and we prove that in this case the problem can also be solved in polynomial time.

4 An extremal problem for clique complexes

Remember that $\mu(G)$ is the minimum number of matroids which we need for the representation of the clique complex of G as their intersection. Furthermore, let $\mu(n)$ be the maximum of $\mu(G)$ over all graphs G with n vertices. Namely,

$$\mu(n) := \max\{\mu(G) \mid G \text{ has } n \text{ vertices}\}.$$

In this section, we determine $\mu(n)$ exactly. It is straightforward to observe that $\mu(1) = 1$. For the case of $n \geq 2$, we can immediately obtain $\mu(n) \leq \binom{n}{2}$ from Lemmas 2.2 and 3.1. However, the following theorem tells us that the truth is in fact much better.



$K_1 \cup K_5$

Figure 3: The graph $K_1 \cup K_5$.

Theorem 4.1. *For every $n \geq 2$, it holds that $\mu(n) = n - 1$.*

First, we prove that $\mu(n) \geq n - 1$. Consider the graph $K_1 \cup K_{n-1}$. (Figure 3 shows $K_1 \cup K_5$.)

Lemma 4.2. *For $n \geq 2$, we have that $\mu(K_1 \cup K_{n-1}) = n - 1$. Particularly it follows that $\mu(n) \geq n - 1$.*

Proof. First, observe that $\overline{K_1 \cup K_{n-1}}$ has $n - 1$ edges. Therefore, Lemma 3.1 implies that the number of the circuits of $\mathfrak{C}(K_1 \cup K_{n-1})$ is $n - 1$. Then, by the argument below the proof of Lemma 2.2, it follows that $\mu(K_1 \cup K_{n-1}) \leq |\mathcal{C}(\mathfrak{C}(K_1 \cup K_{n-1}))| = n - 1$.

Now, suppose that $\mu(K_1 \cup K_{n-1}) \leq n - 2$. By Theorem 3.3, there exist at most $n - 2$ stable-set partitions $\mathcal{P}^{(1)}, \dots, \mathcal{P}^{(n-2)}$ of $K_1 \cup K_{n-1}$ satisfying Condition P, namely, each edge e of $\overline{K_1 \cup K_{n-1}}$ is contained in some set $S \in \bigcup_{i=1}^{n-2} \mathcal{P}^{(i)}$. Then, the pigeon hole principle tells us that there exists an index $i^* \in \{1, \dots, n-2\}$ such that at least two edges of $\overline{K_1 \cup K_{n-1}}$ are contained in sets of $\mathcal{P}^{(i^*)}$. Let e, e' be such (distinct) edges of $\overline{K_1 \cup K_{n-1}}$ and $P_e, P_{e'} \in \mathcal{P}^{(i^*)}$ be unique sets such that $e \subseteq P_e$ and $e' \subseteq P_{e'}$. (The uniqueness follows from the fact that $\mathcal{P}^{(i^*)}$ is a partition.) Now, remember that e and e' share a vertex (since e, e' are edges of $\overline{K_1 \cup K_{n-1}}$). This implies that $P_e \cap P_{e'} \neq \emptyset$. Therefore, it holds that $P_e = P_{e'}$ since $\mathcal{P}^{(i^*)}$ is a partition. Set $e = \{u, v\}$ and $e' = \{u, v'\}$. (Here, u is the vertex shared by e and e'.) This implies that $\{v, v'\}$ is also contained in P_e. However, $\{v, v'\}$ is an edge of $K_1 \cup K_{n-1}$. This contradicts the fact that $\mathcal{P}^{(i^*)}$ is a stable-set partition (i.e., P_e is a stable set of $K_1 \cup K_{n-1}$). Thus, it follows that $\mu(K_1 \cup K_{n-1}) = n - 1$.

For the second part, we just follow the definition of $\mu(n)$. Then we conclude that $\mu(n) \geq \mu(K_1 \cup K_{n-1}) = n - 1$. □

Next we prove that $\mu(n) \leq n - 1$. To do that, first we look at the relation of $\mu(G)$ with the edge-chromatic number $\chi'(\overline{G})$ of the complement.

Lemma 4.3. *It holds that $\mu(G) \leq \chi'(\overline{G})$ for every graph G with n vertices. Particularly, if n is even then we have that $\mu(G) \leq n - 1$, and if n is odd then we have that $\mu(G) \leq n$. Moreover, if $\mu(G) = n$ then n is odd and the maximum degree of $\overline{G}$ is $n - 1$ (i.e., G has an isolated vertex).*

Proof. Consider a minimum proper edge-coloring of $\overline{G}$, and let $k = \chi'(\overline{G})$. Now, we construct k stable-set partitions of a graph G with n vertices from this edge-coloring.

We have the color classes $C^{(1)}, \dots, C^{(k)}$ of the edges from the minimum proper edge-coloring. Let us take a color class $C^{(i)} = \{e_1^{(i)}, \dots, e_{l_i}^{(i)}\}$ ($i \in \{1, \dots, k\}$) and construct a stable-set partition $\mathcal{P}^{(i)}$ of G from $C^{(i)}$ as follows: S is a member of $\mathcal{P}^{(i)}$ if and only if either (1) S is a two-element set belonging to $C^{(i)}$ (i.e., $S = e_j^{(i)}$ for some $j \in \{1, \dots, l_i\}$) or (2) S is a one-element set $\{v\}$ which is not used in $C^{(i)}$ (i.e., $v \notin e_j^{(i)}$ for any $j \in \{1, \dots, l_i\}$). Notice that $\mathcal{P}^{(i)}$ is actually a stable-set partition. Then we collect all the stable-set partitions $\mathcal{P}^{(1)}, \dots, \mathcal{P}^{(k)}$ constructed by the procedure above. Moreover, we can check that these stable-set partitions satisfy Condition P in Theorem 3.3 (since each edge of $\overline{G}$ appears in exactly one of the $C^{(i)}$'s). Hence, we have that $\mu(G) \leq k = \chi'(\overline{G})$ by Theorem 3.3. Figure 4 illustrates the construction. In this example, we have that $\chi'(\overline{G}) = 3$. The first row shows a given graph G and its complement $\overline{G}$. In the second row, we can find a minimum proper edge-coloring of $\overline{G}$, and each $C^{(i)}$ depicts a color class of this coloring. The constructed stable-set partitions are put in the third row.

Now, notice that $\chi'(\overline{G}) \leq \chi'(K_n)$ for any graph G with n vertices. Thus, if n is even, then we can conclude that $\mu(G) \leq n - 1$ since $\chi'(K_n) = n - 1$. Similarly, if n is odd, then we can conclude that $\mu(G) \leq n$ since $\chi'(K_n)$ is n.

For the last part of the lemma, assume that $\mu(G) = n$. From the discussion above, n should be odd. Note that Vizing's theorem (see [6] for example) says that for a graph H with maximum degree $\Delta(H)$ we have that $\chi'(H) = \Delta(H)$ or $\Delta(H) + 1$. Since $\Delta(\overline{G}) \leq n - 1$, we have that $\mu(G) \leq \chi'(\overline{G}) \leq \Delta(\overline{G}) + 1 \leq n$. Therefore, $\mu(G) = n$ holds only if $\Delta(\overline{G}) + 1 = n$. □

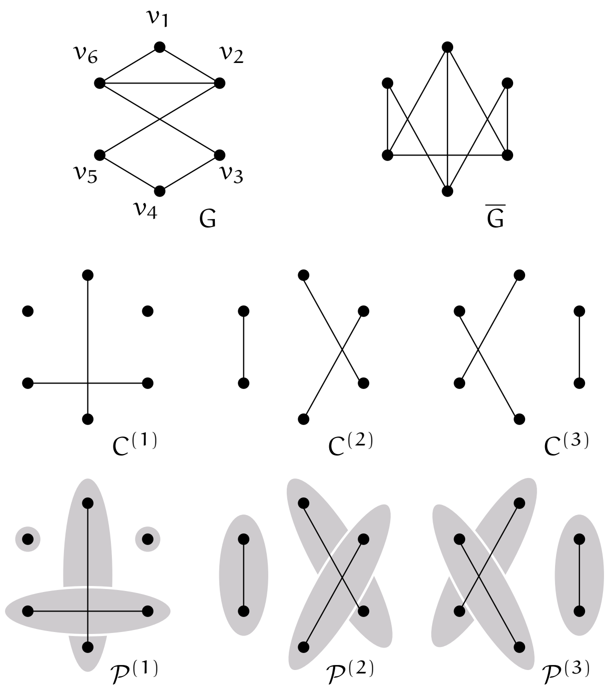


Figure 4: The construction in the proof of Lemma 4.3.

Now, we show that if a graph G with n vertices (where n is odd) has an isolated vertex then $\mu(G) \leq n-1$. This completes the proof of Theorem 4.1.

Lemma 4.4. *Let* n *be odd and* G *be a graph with* n *vertices which has an isolated vertex. Then it holds that* $\mu(G) \leq n-1$.

Proof. Let v^* be an isolated vertex of G. Consider the subgraph of G induced by $V(G) \setminus \{v^*\}$. Denote this induced subgraph by G' (i.e., $G' = G[V(G) \setminus \{v^*\}]$). Since G' has $n-1$ vertices, which is even, we have $\mu(G') \leq n-2$ from Lemma 4.3.

Now we construct $n-1$ stable-set partitions of G which satisfy Condition P from $n-2$ stable-set partitions of G' which also satisfy Condition P. Denote the vertices of G' by $v_1, \ldots, v_{n-1}$, and stable-set partitions of G' satisfying Condition P by $\mathcal{P}'^{(1)}, \ldots, \mathcal{P}'^{(n-2)}$ (where some of them may be identical in case $\mu(G') < n-2$). Then construct stable-set partitions $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(n-2)}, \mathcal{P}^{(n-1)}$ of G as follows. For each $i \in \{1, \ldots, n-2\}$, put $P \in \mathcal{P}^{(i)}$ if and only if either (1) $P \in \mathcal{P}'^{(i)}$ and $v_i \notin P$ or (2) $v^* \in P$, $P \setminus \{v^*\} \in \mathcal{P}'^{(i)}$ and $v_i \in P$. Furthermore, put $P \in \mathcal{P}^{(n-1)}$ if and only if either (1) $P = \{v_i\}$ ($i \in \{1, \ldots, n-2\}$) or (2) $P = \{v^*, v_{n-1}\}$. Figure 5 illustrates the construction of $\mathcal{P}^{(i)}$ ($i \in \{1, \ldots, n-1\}$). The first row shows a given graph G where the topmost vertex v^* is isolated. In the second row, we can find three stable-set partitions of $G' = G[\{v_1, v_2, v_3, v_4\}]$ satisfying Condition P. In this row, the symbol $\circ$ is used for the indication of the neglected vertex v^*. In the third row (lowest), the constructed stable-set partitions of G are shown according to the considered vertices.

For conclusion, it is enough to check that the stable-set partitions $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(n-1)}$ constructed above satisfy Condition P. Choose any edge e of $\overline{G}$. If e is also an edge of $\overline{G'}$, then we can find a set $S' \in \bigcup_{i=1}^{n-2} \mathcal{P}'^{(i)}$ such that $e \subseteq S'$ since $\mathcal{P}'^{(1)}, \ldots, \mathcal{P}'^{(n-2)}$ satisfy Condition P. From the construction of $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(n-2)}$, we can observe that for each $i \in \{1, \ldots, n-2\}$ and each $P' \in \mathcal{P}'^{(i)}$ there exists a set $P \in \mathcal{P}^{(i)}$ such that $P' \subseteq P$. Therefore, for S' above, we also have $S \in \bigcup_{i=1}^{n-2} \mathcal{P}^{(i)}$ such that $S' \subseteq S$, which implies that $e \subseteq S$. If e is not an edge of $\overline{G'}$, then e has a form as $e = \{v^*, v_i\}$ for some $i \in \{1, \ldots, n-1\}$. Then it turns out that e is contained in a member of $\mathcal{P}^{(i)}$ which was put in $\mathcal{P}^{(i)}$ due to the condition (2). In this way, we have verified that $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(n-1)}$ satisfy Condition P. □

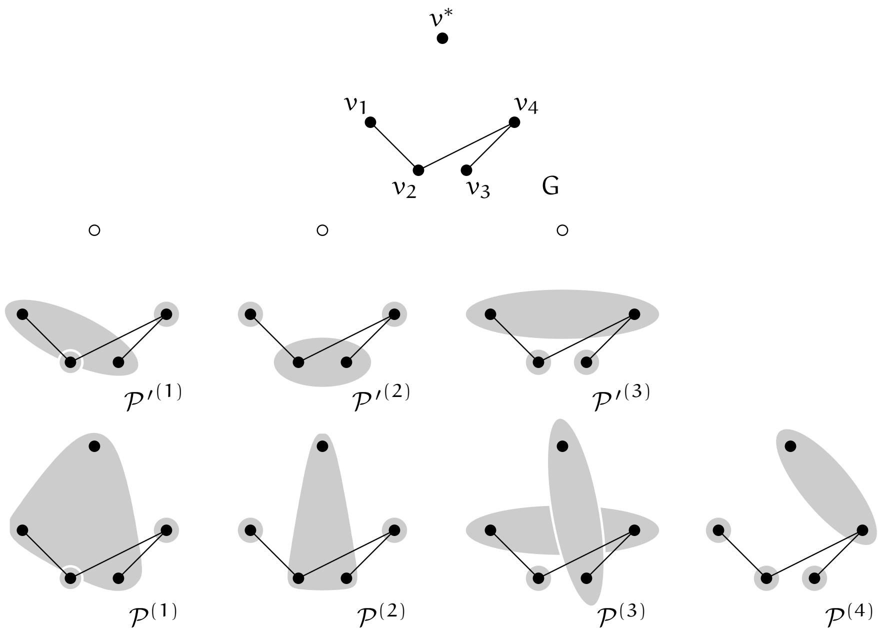


Figure 5: The construction of stable-set partitions from an edge-coloring.

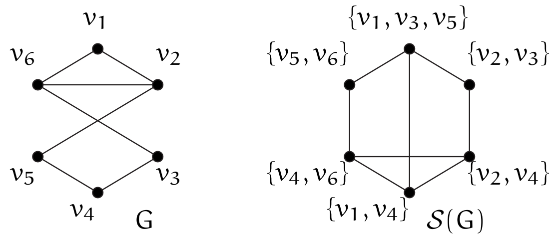


Figure 6: An example of stable-set graphs.

5 Characterizations for two matroids

In this section, we look more closely at a clique complex which can be represented as the intersection of two matroids. Note that Fekete, Firla & Spille [9] gave a characterization of the graphs whose matching complexes can be represented as the intersections of two matroids. So the theorem in this section is a generalization of their result. (Their result will be discussed in Section 7.)

To do this, we invoke another concept. The *stable-set graph* of a graph $G = (V, E)$ is a graph whose vertices are the maximal stable sets of G and two vertices of which are adjacent if and only if the corresponding two maximal stable sets of G share a vertex in G. We denote the stable-set graph of a graph G by $\mathcal{S}(G)$. Figure 6 is an example of stable-set graphs.

The next lemma establishes the relationship between $\mu(G)$ and the chromatic number $\chi(\mathcal{S}(G))$ of the stable-set graph.

Lemma 5.1. *Let* G *be a graph. If the stable-set graph* $\mathcal{S}(G)$ *is* k*-colorable, then the clique complex* $\mathfrak{C}(G)$ *can be represented as the intersection of* k *matroids. In other words, it holds that* $\mu(G) \leq \chi(\mathcal{S}(G))$.

Proof. Assume that we are given a proper k-coloring c of $\mathcal{S}(G)$, i.e., $c : V(\mathcal{S}(G)) \to \{1, \ldots, k\}$ where $c(S) \neq c(T)$ if $S \cap T \neq \emptyset$. Then gather the maximal stable sets of G which have the same color with respect to the coloring c, that is, put $C_i = \{S \in V(\mathcal{S}(G)) \mid c(S) = i\}$ for each $i \in \{1, \ldots, k\}$. We can see that the members of C_i are disjoint maximal

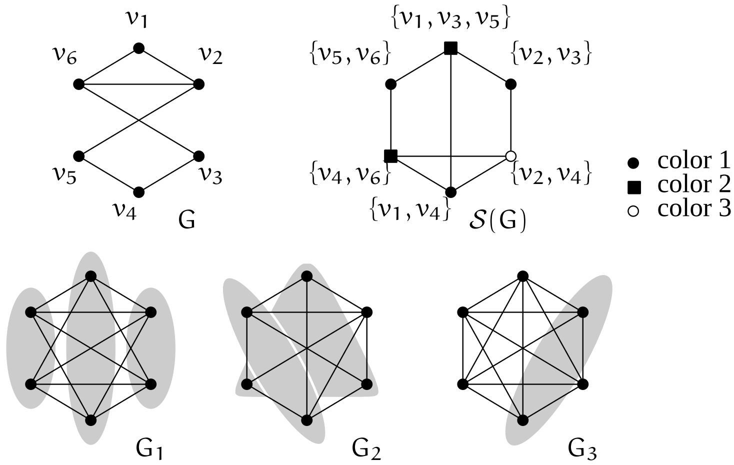


Figure 7: The construction of G_i in the proof of Lemma 5.1.

stable sets of G for each $i \in \{1, \ldots, k\}$.

Now we construct a graph G_i from C_i as follows. The vertex set of G_i is the same as that of G, and two vertices of G_i are adjacent if and only if either (1) one belongs to a maximal stable set in C_i and the other belongs to another maximal stable set in C_i, or (2) one belongs to a maximal stable set in C_i and the other belongs to no maximal stable set in C_i. Figure 7 explains the construction of G_i. In Figure 7, three colors of $\mathcal{S}(G)$ are depicted by •, ■ and ○, and in the second row, the shaded groups show maximal independent sets corresponding to the vertices in $\mathcal{S}(G)$ colored by the identical colors.

Note that G_i is complete r-partite, where r is equal to $|C_i|$ plus the number of the vertices which do not belong to any maximal stable set in C_i. (This holds in general, not just in the picture above.) Then consider $\mathfrak{C}(G_i)$, the clique complex of G_i. By Lemma 3.2, we can see that $\mathfrak{C}(G_i)$ is actually a matroid. Since an edge of G is also an edge of G_i (or by Lemma 3.5), we have that $\mathfrak{C}(G) \subseteq \mathfrak{C}(G_i)$.

Now we consider the intersection $\mathcal{I} = \bigcap_{i=1}^{k} \mathfrak{C}(G_i)$. Since $\mathfrak{C}(G) \subseteq \mathfrak{C}(G_i)$ for every $i \in \{1, \ldots, k\}$, we have $\mathfrak{C}(G) \subseteq \mathcal{I}$. Since each circuit of $\mathfrak{C}(G)$ is also a circuit of $\mathfrak{C}(G_i)$ for some $i \in \{1, \ldots, k\}$ (recall Lemma 3.1), we also have $\mathcal{C}(\mathfrak{C}(G)) \subseteq \mathcal{C}(\mathcal{I})$, which implies $\mathfrak{C}(G) \supseteq \mathcal{I}$. Thus we have $\mathfrak{C}(G) = \mathcal{I}$. □

Note that the converse of Lemma 5.1 does not hold in general even if $k = 3$. A counterexample is the graph $G = (V, E)$ defined as $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}\}$. See Figure 8. In the graph shown in Figure 8, consider the following stable-set partitions of G:

$$\mathcal{P}^{(1)} = \{\{v_1, v_3, v_5\}, \{v_2, v_4, v_6\}\},$$
$$\mathcal{P}^{(2)} = \{\{v_1, v_3, v_6\}, \{v_2, v_4, v_5\}\},$$
$$\mathcal{P}^{(3)} = \{\{v_1, v_4, v_5\}, \{v_2, v_3, v_6\}\}.$$

We can check that these stable-set partitions fulfill Condition P in Theorem 3.3. Therefore, by Theorem 3.3, we can see that $\mathfrak{C}(G)$ is the intersection of three partition matroids $\mathcal{I}(\mathcal{P}^{(1)})$, $\mathcal{I}(\mathcal{P}^{(2)})$ and $\mathcal{I}(\mathcal{P}^{(3)})$. However, $\mathcal{S}(G)$ is not 3-colorable but 4-colorable. (In Figure 8, a proper 4-coloring of $\mathcal{S}(G)$ is also indicated.)

By a similar argument, we can also see that, if we consider a graph G consisting of $n/2$ independent edges only (i.e. a graph itself being a perfect matching), then $\mu(G) = \Theta(n)$ and $\chi(\mathcal{S}(G)) = \Theta(2^{n/2})$. Therefore, the difference between $\mu(G)$ and $\chi(\mathcal{S}(G))$ can be arbitrarily large.

However, the converse holds if $k = 2$.

Theorem 5.2. *Let* G *be a graph. The clique complex* $\mathfrak{C}(G)$ *can be represented as the intersection of two matroids if and only if the stable-set graph* $\mathcal{S}(G)$ *is 2-colorable (i.e., bipartite).*

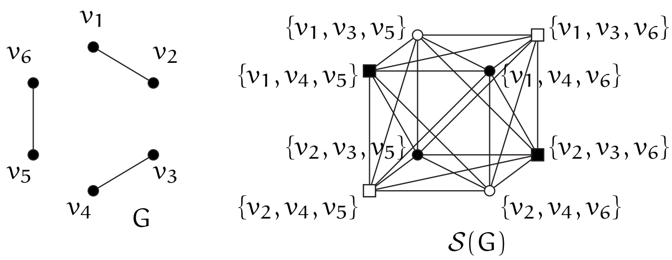


Figure 8: A counterexample for the converse of Lemma 5.1.

Proof. The if-part is straightforward from Lemma 5.1. Now we prove the only-if-part. Assume that $\mathfrak{C}(G)$ is represented as the intersection of two matroids. Due to Theorem 3.3, we may assume that these two matroids are associated with stable-set partitions $\mathcal{P}^{(1)}, \mathcal{P}^{(2)}$ of G satisfying Condition P.

Let S be a maximal stable set of G. Now we claim that $S \in \mathcal{P}^{(1)} \cup \mathcal{P}^{(2)}$. To prove this claim, from the maximality of S, we only have to show that $S \subseteq P$ for some $P \in \mathcal{P}^{(1)} \cup \mathcal{P}^{(2)}$. (Then, the maximality of S tells us that $S = P$.) Since $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ are partitions of $V(G)$, this claim clearly holds if $|S| = 1$. If $|S| = 2$, the claim holds from Condition P.

Assume that $|S| \geq 3$. Then consider the following independence system:

$$\mathcal{I} = \{I \subseteq S \mid I \subseteq P \text{ for some } P \in \mathcal{P}^{(1)} \cup \mathcal{P}^{(2)}\}.$$

Choose a base B of $\mathcal{I}$ arbitrarily. Since $B \subseteq S$ and S is a stable set of G, we can see that B is also a stable set of G. This means that B is a dependent set of $\mathfrak{C}(G)$. Therefore, B contains a circuit of $\mathfrak{C}(G)$. By Lemma 3.1, we have that $|B| \geq 2$. If $S = B$ holds then we are done (since $B \subseteq P$ for some $P \in \mathcal{P}^{(1)} \cup \mathcal{P}^{(2)}$). Since $B \subseteq S$, it suffices to show that $B \supseteq S$.

Now, suppose that $S \setminus B \neq \emptyset$ for a contradiction. Pick $u \in S \setminus B$ arbitrarily. Then $\{u, v\}$ is a circuit of $\mathfrak{C}(G)$ for any $v \in B$ since S is a stable set of G and $\{u, v\} \subseteq S$. Without loss of generality, we may assume that $B \subseteq P$ for some $P \in \mathcal{P}^{(1)}$. Then it holds that $\{u\} \cup B \not\subseteq P$. (otherwise, it would violate the maximality of B in $\mathcal{I}$). Therefore, from Condition P, we can see that there should exist some $P' \in \mathcal{P}^{(2)}$ such that $\{u, v\} \subseteq P'$ for all $v \in B$. This implies that $\{u\} \cup B \subseteq P'$, which is a contradiction to the maximality of B. Hence it follows that $S = B$. Thus, the claim is verified.

Now we color the vertices of $\mathcal{S}(G)$, i.e., the maximal stable sets of G, according to $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$. If a maximal stable set S belongs to $\mathcal{P}^{(1)}$, then S is colored by 1. Similarly, if S belongs to $\mathcal{P}^{(2)}$, then S is colored by 2. (If S belongs to both, then S can be colored by either 1 or 2 arbitrarily.) This coloring certainly provides a proper 2-coloring of $\mathcal{S}(G)$ since $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ are partitions of $V(G)$. □

Figure 9 is an illustration of what we saw in the proof. The graph G in Figure 9 has three maximal stable sets, and they form the vertex set of the stable-set graph $\mathcal{S}(G)$. In the second row, we can see two stable-set partitions satisfying Condition P. According to these stable-set partitions, we can color the vertices in $\mathcal{S}(G)$. In this example, $\{v_1, v_3, v_5\}$ is colored by $\bullet$ (color 1) since $\{v_1, v_3, v_5\}$ appears in $\mathcal{P}^{(1)}$, and $\{v_5, v_6\}$ is colored by $\circ$ (color 2) since $\{v_5, v_6\}$ appears in $\mathcal{P}^{(2)}$. Then, $\{v_2, v_4\}$ appears in both of $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$. Therefore we can color it by either $\bullet$ or $\circ$ arbitrarily. In the picture above, we just chose $\circ$.

Some researchers already noticed that the bipartiteness of $\mathcal{S}(G)$ is characterized by other properties. We gather them in the following proposition. Here, the *line graph* of a multigraph G is a graph $L(G)$ such that the vertex set of $L(G)$ is the edge set of G and two vertices in $L(G)$ are adjacent through an edge if and only if the corresponding two edges in G share a vertex in G.

Proposition 5.3. *Let* G *be a graph. Then the following are equivalent.*

(1) *The stable-set graph* $\mathcal{S}(G)$ *is bipartite.*

(2) G *is the complement of the line graph of a bipartite multigraph.*

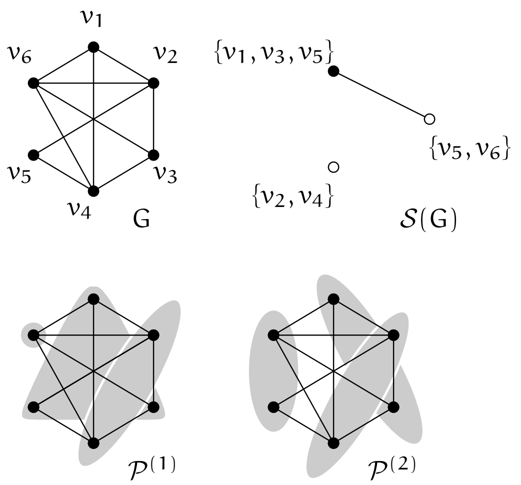


Figure 9: An illustration of the proof of Theorem 5.2.

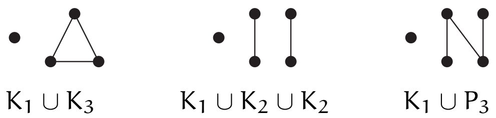


Figure 10: The forbidden induced subgraphs for Proposition 5.3.

(3) G *has no induced subgraph isomorphic to* $K_1 \cup K_3$, $K_1 \cup K_2 \cup K_2$, $K_1 \cup P_3$ *or* $\overline{C_{2k+3}}$ *(*$k = 1, 2, \ldots$*). See Figure 10.*

Proof. The equivalence "(1) ⇔ (2)" is immediate from a result by Cai, Corneil & Proskurowski [3]. Also, the equivalence "(1) ⇔ (3)" is immediate from a result by Protti & Szwarcfiter [23]. □

Note that we can decide whether the stable-set graph of a graph is bipartite or not in polynomial time using the algorithm described by Protti & Szwarcfiter [23]. Here, we mention their algorithm in short. To establish their algorithm, first we have to observe that if $\mathcal{S}(G)$ is bipartite then G contains at most $2n$ maximal stable sets. (This is not trivial. For a proof, see the original paper [23].) Using this observation, they provided the following algorithm. At the first step, we list up the maximal stable sets of G using an algorithm with polynomial delay by Tsukiyama, Ide, Ariyoshi & Shirakawa [26], for example. If the algorithm starts to generate more than $2n$ maximal stable sets then we stop the algorithm and answer "NO" (since $\mathcal{S}(G)$ cannot be bipartite from the observation above). If it generates at most $2n$ maximal stable sets, then we proceed to the second step. At the second step, we explicitly construct $\mathcal{S}(G)$, which can be done in polynomial time since the number of vertices of $\mathcal{S}(G)$ is at most $2n$. Then, as the third step, we check that $\mathcal{S}(G)$ is bipartite or not, which can also be done in polynomial time. If it is bipartite then answer "YES," otherwise "NO." In total, this procedure runs in polynomial time.

As for the maximum weighted clique problem, for the class of graphs satisfying the conditions in Proposition 5.3 we can solve the maximum weighted clique problem exactly in polynomial time by Frank's algorithm [10] for the maximum weighted base problem in the intersection of two matroids. Notice that in Frank's algorithm we need to have a description of the two matroids. However, since the above algorithm by Protti & Szwarcfiter [23] explicitly gives a proper 2-coloring of the stable-set graph if the answer is "YES," from the argument in the proof of Theorem 5.2 we can find the corresponding stable-set partitions of the graph, which are sufficient for running Frank's algorithm.

Speaking of the case of three matroids, we leave the complexity of deciding whether a clique complex is the intersection of three matroids as an open problem. As for the maximum weighted clique problem, the problem of finding a maximum weighted clique in a graph whose clique complex is the intersection of three matroids turns out to be NP-hard, even for the unweighted case. Here, we want to describe the reason briefly. In Corollary 3.7, we mentioned that the class of clique complexes which are the intersections of three matroids is the same as the class of the intersections of three partition matroids. Therefore, our problem is nothing but finding a maximum weighted base in the intersection of three partition matroids. However this problem contains the maximum 3-dimensional matching problem as a special case, which is known to be NP-hard [11] (and even MAX-SNP-hard [15]). That is why our problem is intractable for three matroids.

6 Graphs as independence systems and the intersection of matroids

We can regard a graph as an independence system such that a subset of the vertex set is independent if and only if it is either (1) the empty set, (2) a vertex of the graph or (3) an edge of the graph. In this section we consider how many matroids we need for the representation of a graph (as an independence system) by their intersection. First, we establish a lemma on the matroidal case.

Lemma 6.1. *Let G be a graph. Then the following are equivalent.*

(1) G *is a matroid.*

(2) $\mathfrak{C}(G)$ *is a matroid.*

(3) G *is complete* r*-partite for some* r.

For the proof, we need a truncation. Let $\mathcal{I}$ be an independence system on V. For $k \geq 0$, the k*-th truncation* of $\mathcal{I}$ is the subfamily $\mathcal{I}^{\leq k}$ of $\mathcal{I}$ defined as $\mathcal{I}^{\leq k} = \{X \in \mathcal{I} \mid |X| \leq k\}$. We can see that the truncation of an independence system $\mathcal{I}$ is also an independence system, and if $\mathcal{I}$ is a matroid then $\mathcal{I}^{\leq k}$ is also a matroid for every $k \geq 0$. Note that the k-th truncation is also called the $(k-1)$*-skeleton*, especially in some papers which study "simplicial complexes" instead of "independence systems."

Proof of Lemma 6.1. "(2) ⇔ (3)" is precisely Lemma 3.2. "(2) ⇒ (1)" is immediate from the facts that G is the 2-truncation of $\mathfrak{C}(G)$ and that the truncation of a matroid is also a matroid. Now we prove "(1) ⇒ (3)." Suppose that G is not complete r-partite for any r. Then, G has three vertices u, v, w such that $\{u, v\}$ is an edge but neither $\{u, w\}$ nor $\{v, w\}$ is an edge of G. However, since $\{u, v\}$ and $\{w\}$ are independent sets, by the augmentation axiom $\{u, w\}$ or $\{v, w\}$ should be an edge of G. This is a contradiction. □

The following theorem says that the minimum number of matroids for a graph is the same as that for the clique complex of this graph.

Theorem 6.2. *Let G be a graph. Then G can be represented as the intersection of k matroids if and only if the clique complex $\mathfrak{C}(G)$ can be represented as the intersection of k matroids.*

Proof. First, we show that if the clique complex $\mathfrak{C}(G)$ is the intersection of k matroids then G can be represented as the intersection of k matroids.

Let $\mathfrak{C}(G)$ be represented as the intersection of the matroids $\mathcal{I}_1, \ldots, \mathcal{I}_k$, i.e., $\mathfrak{C}(G) = \bigcap_{i=1}^k \mathcal{I}_i$. Due to Theorem 3.3, without loss of generality, we may assume that $\mathcal{I}_i$ is a partition matroid for each $i \in \{1, \ldots, k\}$. Then consider the truncations $\mathcal{I}_1^{\leq 2}, \ldots, \mathcal{I}_k^{\leq 2}$, and observe that $\bigcap_{i=1}^k \mathcal{I}_i^{\leq 2} = (\bigcap_{i=1}^k \mathcal{I}_i)^{\leq 2}$. On the other hand, we have that $G = \mathfrak{C}(G)^{\leq 2} = (\bigcap_{i=1}^k \mathcal{I}_i)^{\leq 2}$. Thus we conclude that $G = (\bigcap_{i=1}^k \mathcal{I}_i)^{\leq 2}$.

Next we show that if G can be represented as the intersection of k matroids then $\mathfrak{C}(G)$ can also be represented as the intersection of k matroids.

Let G be represented as the intersection of the matroids $\mathcal{J}_1, \ldots, \mathcal{J}_k$, namely $G = \bigcap_{i=1}^k \mathcal{J}_i$. Without loss of generality, we may assume that the size of every base of $\mathcal{J}_i$ is at most two for each $i \in \{1, \ldots, k\}$. (If not, then consider the truncation $\mathcal{J}_i^{\leq 2}$, which does not change the intersection that we are considering since the size of every base in G is at most two.) Then we can regard $\mathcal{J}_i$ as a graph for each $i \in \{1, \ldots, k\}$. Let us denote this graph by G'_i. From Lemma 6.1, the clique complex of G'_i is a matroid (since G'_i is a matroid). Now we have that $G = \bigcap_{i=1}^k G'_i$. Therefore, it holds that $\mathfrak{C}(G) = \mathfrak{C}(\bigcap_{i=1}^k G'_i) = \bigcap_{i=1}^k \mathfrak{C}(G'_i)$. Since we have just observed that $\mathfrak{C}(G'_i)$ is a matroid for each $i \in \{1, \ldots, k\}$, this completes the proof. □

7 Matching complexes

In this section, we apply our theorems to the matching complexes of graphs, and observe that some results by Fekete, Firla & Spille [9] can be obtained from our more general theorems.

A *matching* of a graph $G = (V, E)$ is a subset $M \subseteq E$ of the edge set in which the edges are pairwise disjoint, that is, $e \cap e' = \emptyset$ for each $e, e' \in M$. A *matching complex* of a graph G is the family of matchings of G, and denoted by $\mathfrak{M}(G)$. We can see that the matching complex $\mathfrak{M}(G)$ is indeed an independence system on E. Note that the matching complex $\mathfrak{M}(G)$ is identical to the clique complex of the complement of the line graph of G, i.e., $\mathfrak{M}(G) = \mathfrak{C}(\overline{L(G)})$. Recall that the *line graph* of a graph G is a graph $L(G)$ such that the vertex set of $L(G)$ is the edge set of G and two vertices in $L(G)$ are adjacent through an edge if and only if the corresponding two edges in G share a vertex in G. We also call a graph G a *line graph* if there exists some graph whose line graph is G. For a line graph G, a graph H is called a *root graph* of the line graph G if $G = L(H)$. Note that a root graph of a line graph is not unique in general. For example, K_3 is the line graph of K_3 and also of $K_{1,3}$, i.e., both K_3 and $K_{1,3}$ are the root graphs of K_3. Also, note that not every graph is a line graph; for example, $K_{1,3}$ is not a line graph.

First, let us deduce the characterization of matroidal matching complexes from Lemma 3.2.

Corollary 7.1. *Let G be a graph. The matching complex $\mathfrak{M}(G)$ is a matroid if and only if G is a disjoint union of stars and triangles.*

Proof. Assume that $\mathfrak{M}(G)$ is a matroid. Since $\mathfrak{M}(G) = \mathfrak{C}(\overline{L(G)})$ holds, we have that $\overline{L(G)}$ is a complete r-partite graph for some r by Lemma 3.2. This means that $L(G)$ is a disjoint union of complete graphs. Let K be a connected component of $L(G)$, which is a complete graph. Now, we want to find the root graphs of K. Then we can observe that the root graph of K_1 is $K_2 (= K_{1,1})$, and this is a unique root graph of K_1; the root graph of K_2 is $K_{1,2}$, and this is a unique root graph of K_2; the root graphs of K_3 are K_3 and $K_{1,3}$, and they are the only root graphs of K_3; the root graph of K_n ($n \geq 4$) is $K_{1,n}$, and this is a unique root graph of K_n. (Note that our graph is always simple, i.e., without a loop or a multiple edge.) Therefore, G is a disjoint union of stars and triangles.

Let us show the converse. Assume that G is a disjoint union of stars and triangles. Then we can see that $\overline{L(G)}$ is a complete multipartite graph. From Lemma 3.2, it follows that $\mathfrak{M}(G) = \mathfrak{C}(\overline{L(G)})$ is a matroid. □

Fekete, Firla & Spille [9] studied the matching complex in the same spirit as we did in this paper. They proved the following statement for the intersection of two matroids. In this paper, we derive this result as a corollary from our theorem.

Corollary 7.2 ([9]). *Let G be a graph. The matching complex $\mathfrak{M}(G)$ is the intersection of two matroids if and only if G contains no subgraph (not necessarily induced) isomorphic to C_{2k+3} ($k = 1, 2, \ldots$), and each triangle in G has at most one vertex of degree more than two.*

To prove Corollary 7.2, we use the fact on a line graph.

Lemma 7.3. *Let G be a graph, H be a line graph, and $R_1, \ldots, R_k$ be the root graphs of H. Then $L(G)$ contains no induced subgraph isomorphic to H if and only if G contains no subgraph (not necessarily induced) isomorphic to any of $R_1, R_2, \ldots, R_k$.*

Proof. Straightforward from the definitions of a line graph and a root graph. □

With use of Lemma 7.3, we can prove Corollary 7.2.

Proof of Corollary 7.2. Assume that there exist two matroids $\mathcal{I}_1, \mathcal{I}_2$ on $E(G)$ such that $\mathfrak{M}(G) = \mathcal{I}_1 \cap \mathcal{I}_2$. From the observation above, this is equivalent to that $\mathfrak{C}(\overline{L(G)}) = \mathcal{I}_1 \cap \mathcal{I}_2$. By Theorem 5.2, this is also equivalent to that $\overline{L(G)}$ contains no induced subgraph isomorphic to $K_1 \cup K_3$, $K_1 \cup K_2 \cup K_2$, $K_1 \cup P_3$ or $\overline{C_{2k+3}}$ ($k = 1, 2, \ldots$). Therefore, by Lemma 7.3, we can see that this is also equivalent to that $L(G)$ contains no subgraph (not necessarily induced) isomorphic to $K_{1,3} = \overline{K_1 \cup K_3}$, $W_4 = \overline{K_1 \cup K_2 \cup K_2}$, $W_4^- = \overline{K_1 \cup P_3}$ or C_{2k+3} ($k = 1, 2, \ldots$). See Figure 11 for the shapes of these graphs.

Now, we want to know the root graphs of $K_{1,3}$, W_4, W_4^-, and C_{2k+3} ($k = 1, 2, \ldots$). Then we can observe the following. (1) There is no root graph of $K_{1,3}$ (i.e., $K_{1,3}$ is not a line graph). (2) The root graph of W_4 is C_4^+ (in the

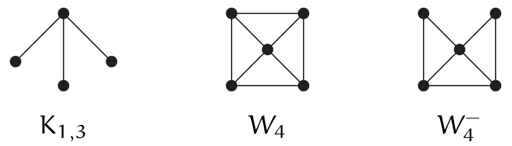


Figure 11: Graphs appearing in the proof of Corollary 7.2.

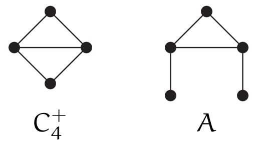


Figure 12: The root graphs appearing in the proof of Corollary 7.2.

picture below) and this is a unique root graph of W_4. (3) The root graph of W_4^- is A (in the picture below) and this is a unique root graph of W_4^-. (4) For each $k = 1, 2, \ldots$, the root graph of C_{2k+3} is C_{2k+3} and this is a unique root graph of C_{2k+3}. See Figure 12.

Thus, we can see that Lemma 7.3 implies that the matching complex $\mathfrak{M}(G)$ is the intersection of two matroids if and only if G contains no subgraph isomorphic to C_4^+, A or C_{2k+3} ($k = 1, 2, \ldots$). Hence, for the proof of the corollary, it is enough to observe that G contains no subgraph isomorphic to C_4^+ or A if and only if each triangle in G has at most one vertex of degree more than two.

To observe that, first assume that G contains no subgraph isomorphic to C_4^+ or A and also suppose that there exists a triangle in G which has at least two vertices of degree more than two. Let u and v be such vertices in the triangle ($u \neq v$). Then the above assumption means that there exist edges $\{u, x\}$ and $\{v, y\}$ in G. In case $x = y$, we can see that G contains C_4^+ as a subgraph. In case $x \neq y$, we can see that G contains A as a subgraph. Therefore, in both cases this is a contradiction.

Conversely, assume that each triangle in G has at most one vertex of degree more than two. Pick a triangle T in G arbitrarily. Then we can see that T cannot be contained in a subgraph isomorphic to C_4^+ or A in G since C_4^+ and A have two vertices of degree more than two. This means that G contains no subgraph isomorphic to C_4^+ or A. This concludes the proof. □

Fekete, Firla & Spille [9] also gave a characterization of the matching complex which can be represented as the intersection of k matroids for a general k. Their characterization involves an integer programming formulation of the problem to find the right k. We observe that their characterization is also a corollary of our theorem. To do that, we need to introduce their formulation.

First, we introduce the variables in the formulation. Since the circuits of $\mathfrak{M}(G)$ are the paths of length 2 (this is an immediate consequence from Lemma 3.1 and the fact that $\mathfrak{M}(G) = \mathfrak{C}(\overline{L(G)})$), it makes sense that we use a variable $\mathbf{x} \in \{0, 1\}^{\{1,\ldots,k\}\times\mathcal{P}(G)}$ where $\mathcal{P}(G)$ is the family of all paths of length 2 in G. We denote a path of length 2 in G by (u, v, w) when v is the midpoint of the path and u, w are the endpoints of the path. Note that the path (w, v, u) is identified with (u, v, w). The interpretation of the variable $\mathbf{x}$ is as follows. Assume that $\mathfrak{M}(G)$ is the intersection of matroids $\mathcal{I}_1, \ldots, \mathcal{I}_k$. For $i \in \{1, \ldots, k\}$ and $(u, v, w) \in \mathcal{P}(G)$, $x[i, (u, v, w)] = 1$ if (u, v, w) is a circuit of $\mathcal{I}_i$; otherwise $x[i, (u, v, w)] = 0$. Then Fekete, Firla & Spille [9] considered the following set of constraints.

Cover condition: $\sum_{i=1}^{k} x[i, (u, v, w)] \geq 1$ for all $(u, v, w) \in \mathcal{P}(G)$,

Claw condition: $x[i, (u, v, w)] + x[i, (u, v, t)] + x[i, (w, v, t)] \neq 2$ for all $i \in \{1, \ldots, k\}$ and $(u, v, w), (u, v, t), (w, v, t) \in \mathcal{P}(G)$,

Triangle condition: $x[i, (u, v, w)] + x[i, (v, w, u)] + x[i, (w, u, v)] \neq 2$ for all $i \in \{1, \ldots, k\}$ and $(u, v, w), (v, w, u), (w, u, v) \in \mathcal{P}(G)$,

Matching condition: $x[i, (u, v, w)] + x[i, (v, w, t)] \leq 1$ for all $i \in \{1, \ldots, k\}$ and (u, v, w), $(v, w, t) \in \mathcal{P}(G)$.

(See Fekete, Firla & Spille [9] for the detail of these constraints.) Note that Claw condition and Triangle condition can be written as linear inequality constraints as well.

Corollary 7.4 ([9]). *Let* G *be a graph. Then* $\mathfrak{M}(G)$ *is the intersection of* k *matroids if and only if there exists a vector* $x \in \{0,1\}^{\{1,\dots,k\}\times\mathcal{P}(G)}$ *which satisfies all of the four conditions above (namely, Cover condition, Claw condition, Triangle condition and Matching condition).*

Proof. Let $G = (V, E)$ be a given graph. First, let us assume that $\mathfrak{M}(G) = \mathfrak{C}(\overline{L(G)})$ is the intersection of k matroids. Then, by Theorem 3.3, there exist k stable-set partitions $\mathcal{P}^{(1)}, \dots, \mathcal{P}^{(k)}$ of $\overline{L(G)}$ which satisfy the following condition: $\{e, f\} \in \binom{E}{2}$ is an edge of $L(G)$ if and only if $\{e, f\} \subseteq S$ for some $S \in \bigcup_{i=1}^{k} \mathcal{P}^{(i)}$. Put $e = \{u, v\}$ and $f = \{w, t\}$ for some $u, v, w, t \in V$. Then, we can see that this condition is equivalent to that $\{e, f\} \in \binom{E}{2}$ forms a path $(u, v = t, w)$ of length 2 in G if and only if $\{e, f\} \subseteq S$ for some $S \in \bigcup_{i=1}^{k} \mathcal{P}^{(i)}$. In the sequel, we write "$(u, v, w) \subseteq S$" instead of "$\{e, f\} \subseteq S$" when $e = \{u, v\}$ and $f = \{w, t\}$ form the path $(u, v = t, w)$ of length 2. Let us summarize this condition as follows and call it Condition P'.

<u>Condition P'</u>:

$\{e, f\} \in \binom{E}{2}$ forms a path $(u, v = t, w)$ of length 2 in G (where $e = \{u, v\}$ and $f = \{w, t\}$) if and only if $(u, v, w) \subseteq S$ for some $S \in \bigcup_{i=1}^{k} \mathcal{P}^{(i)}$.

Now, we construct $\overline{x} \in \{0,1\}^{\{1,\dots,k\}\times\mathcal{P}(G)}$ from our stable-set partitions. For $i \in \{1, \dots, k\}$ and $(u, v, w) \in \mathcal{P}(G)$, set $\overline{x}[i, (u, v, w)] = 1$ if $(u, v, w) \subseteq S$ for some $S \in \mathcal{P}^{(i)}$; set $\overline{x}[i, (u, v, w)] = 0$ otherwise. Then, we show that the vector $\overline{x}$ constructed above satisfies the four conditions.

First, check Cover condition. Fix a path (u, v, w) of length 2 in G arbitrarily. Then, from Condition P', there exists at least one index i^* such that $(u, v, w) \subseteq S$ for some $S \in \mathcal{P}^{(i^*)}$. Our construction implies that $\overline{x}[i^*, (u, v, w)] = 1$. Therefore, we have that $\sum_{i=1}^{k} \overline{x}[i, (u, v, w)] \geq 1$. Since this inequality holds for all paths of length 2 in G, we can see that $\overline{x}$ satisfies Cover condition.

Second, we check Claw condition. Suppose that Claw condition is violated, namely there exist an index $i \in \{1, \dots, k\}$ and paths $(u, v, w), (u, v, t), (w, v, t) \in \mathcal{P}(G)$ such that $\overline{x}[i, (u, v, w)] + \overline{x}[i, (u, v, t)] + \overline{x}[i, (w, v, t)] = 2$. By the symmetry of $(u, v, w), (u, v, t), (w, v, t)$, we may assume that $\overline{x}[i, (u, v, w)] = 1$, $\overline{x}[i, (u, v, t)] = 1$ and $\overline{x}[i, (w, v, t)] = 0$ without loss of generality. The construction of $\overline{x}$ implies that there exist $S_{uw}, S_{ut} \in \mathcal{P}^{(i)}$ such that $(u, v, w) \subseteq S_{uw}$ and $(u, v, t) \subseteq S_{ut}$. Therefore, $\{u, v\} \in S_{uw}$ and $\{u, v\} \in S_{ut}$. This means that $S_{uv} \cap S_{ut} \neq \emptyset$. On the other hand, since $\mathcal{P}^{(i)}$ is a partition of E, and $S_{uw}, S_{ut} \in \mathcal{P}^{(i)}$, it holds that $S_{uw} \cap S_{ut} = \emptyset$. So, we have a contradiction. Thus, we have shown that $\overline{x}$ satisfies Claw condition.

Next, we check Triangle condition. Suppose that Triangle condition is violated, i.e., there exist an index $i \in \{1, \dots, k\}$ and paths $(u, v, w), (v, w, u), (w, u, v) \in \mathcal{P}(G)$ such that $\overline{x}[i, (u, v, w)] + \overline{x}[i, (v, w, u)] + \overline{x}[i, (w, u, v)] = 2$. By the symmetry of $(u, v, w), (v, w, u), (w, u, v)$, we may assume that $\overline{x}[i, (u, v, w)] = 1$, $\overline{x}[i, (v, w, u)] = 1$ and $\overline{x}[i, (w, u, v)] = 0$, without loss of generality. Then, our construction implies that there exist $S_u, S_v \in \mathcal{P}^{(i)}$ such that $(u, v, w) \subseteq S_u$ and $(v, w, u) \subseteq S_v$. Therefore, we can see that $\{v, w\} \in S_u$ and $\{v, w\} \in S_v$. This means that $S_u \cap S_v \neq \emptyset$. On the other hand, since $\mathcal{P}^{(i)}$ is a partition of E, and $S_u, S_v \in \mathcal{P}^{(i)}$, it holds that $S_u \cap S_v = \emptyset$. So, they contradict each other. Thus, we have shown that $\overline{x}$ satisfies Triangle condition.

Finally, we check Matching condition. Suppose that Matching condition is violated, i.e., there exist $i \in \{1, \dots, k\}$ and $(u, v, w), (v, w, t) \in \mathcal{P}(G)$ such that $\overline{x}[i, (u, v, w)] + \overline{x}[i, (v, w, t)] > 1$. Since $\overline{x}$ is a $\{0,1\}$-vector, we have that $\overline{x}[i, (u, v, w)] = 1$ and $\overline{x}[i, (v, w, t)] = 1$. Because of our construction, there exist $S_u, S_v \in \mathcal{P}^{(i)}$ such that $(u, v, w) \subseteq S_u$ and $(v, w, t) \subseteq S_v$. Therefore, we can see that $\{v, w\} \in S_u$ and $\{v, w\} \in S_v$. Then, by the same reason as the case for Triangle condition, we obtain a contradiction. Thus, we have checked that $\overline{x}$ meets Matching condition.

As a conclusion of the discussion above, we have obtained the only-if part of the corollary. So it remains to show the if part.

To do that, assume that there exists a vector $x \in \{0,1\}^{\{1,\dots,k\}\times\mathcal{P}(G)}$ which satisfies Cover condition, Claw condition, Triangle condition and Matching condition. From this vector, we construct k stable-set partitions $\mathcal{Q}^{(1)}, \dots, \mathcal{Q}^{(k)}$ of $\overline{L(G)}$ which satisfy Condition P' above. Since Condition P' is equivalent to Condition P in Theorem 3.3, this concludes the proof.

Fix $i \in \{1,\dots,k\}$. Then we put $\{\{u,v\}\} \in \mathcal{Q}^{(i)}$ if there exists no $(u,v,w) \in \mathcal{P}(G)$ such that $x[i,(u,v,w)] = 1$ and also there exists no $(v,u,t) \in \mathcal{P}(G)$ such that $x[i,(v,u,t)] = 1$. Furthermore, we put $\{\{u,v\},\{v,w\}\} \in \mathcal{Q}^{(i)}$ if $x[i,(u,v,w)] = 1$.

Now, we must check that $\mathcal{Q}^{(i)}$ is indeed a stable-set partition of $\overline{L(G)}$ for each $i \in \{1,\dots,k\}$ as desired. Fix $i \in \{1,\dots,k\}$ arbitrarily. First, let us check that $\mathcal{Q}^{(i)}$ is a partition of $V(\overline{L(G)})$, i.e., a partition of $E(G)$. Clearly $E(G) = \bigcup \mathcal{Q}^{(i)}$ for each $i \in \{1,\dots,k\}$. Suppose, for contradiction, that there exist two distinct sets $S,T \in \mathcal{Q}^{(i)}$ such that $S \cap T \neq \emptyset$. Since each set in $\mathcal{Q}^{(i)}$ is of size 1 or 2, we have the following two cases. As the first case, assume that $|S| = 1$ and $|T| = 2$, say $S = \{\{u,v\}\}$ and $T = \{\{u,v\},\{v,w\}\}$. However, this contradicts our construction of $\mathcal{Q}^{(i)}$. The second case is where $|S| = |T| = 2$. We have two subcases. Assume that, say, $S = \{\{u,v\},\{v,w\}\}$ and $T = \{\{u,v\},\{v,t\}\}$ where $w \neq t$. Then from our construction we have that $x[i,(u,v,w)] = 1$ and $x[i,(u,v,t)] = 1$. By Claw condition, we should have $x[i,(t,v,w)] = 1$. However, Matching condition requires $x[i,(u,v,t)] + x[i,(t,v,w)] \leq 1$. This is a contradiction. Next, assume that, say, $S = \{\{u,v\},\{v,w\}\}$ and $T = \{\{v,u\},\{u,t\}\}$. In this case, again from the construction we have that $x[i,(u,v,w)] = 1$ and $x[i,(v,u,t)] = 1$. If $w \neq t$, then this contradicts Matching condition. If $w = t$, then from Triangle condition we should have that $x[i,(u,w,v)] = 1$. However, this again contradicts Matching condition. Thus, $\mathcal{Q}^{(i)}$ partitions $E(G)$.

Secondly, we check that each set $S \in \mathcal{Q}^{(i)}$ is a stable set of $\overline{L(G)}$. If $|S| = 1$, then clearly S is stable. Assume that $|S| = 2$, say $S = \{\{u,v\},\{v,w\}\}$. Since (u,v,w) is a path of length 2 in G, $\{u,v\}$ and $\{v,w\}$ are adjacent in $L(G)$. This means that they are not adjacent in $\overline{L(G)}$. Therefore $\{\{u,v\},\{v,w\}\}$ is stable in $\overline{L(G)}$. Thus, we proved that $\mathcal{Q}^{(i)}$ is a stable-set partition of $\overline{L(G)}$ for each $i \in \{1,\dots,k\}$.

Now, we check the constructed stable-set partitions $\mathcal{Q}^{(1)},\dots,\mathcal{Q}^{(k)}$ satisfy Condition P' above. However, this can be easily checked with Cover condition. This concludes the whole proof. □

8 Concluding remarks

In this paper, motivated by the quality of a natural greedy algorithm for the maximum weighted clique problem, we characterized the number k such that the clique complex of a graph can be represented as the intersection of k matroids (Theorem 3.3). This implies that the problem of determining the clique complex of a given graph has a representation by k matroids or not belongs to NP (Corollary 3.8). Furthermore, in Section 5 we observed that the corresponding problem for two matroids can be solved in polynomial time. However, the problem for three or more matroids is not known to be solved in polynomial time. We leave the further issue on computational complexity of this problem as an open problem. In addition, we showed that $n-1$ matroids are necessary and sufficient for the representation of the clique complexes of all graphs with n vertices (Theorem 4.1), and looked at the relationship between the clique complex of a graph and the graph itself as an independence system (Theorem 6.2).

We proved that the class of clique complexes is the same as the class of the intersections of partition matroids (Corollary 3.7). This result sheds more light on the structure of clique complexes, and may give a new research direction to attack some open problems on them.

Before, Fekete, Firla & Spille [9] studied matching complexes from the viewpoint of matroid intersections. In Section 7, we have observed that some of their results can be derived from our more general theorems.

Finally, we would like to mention open problems arising from the paper. As mentioned at the end of Section 5, we are not aware of a polynomial-time algorithm to decide whether the clique complex of a given graph is the intersection of three matroids or not. This is open. As another open question, we want to mention the following. In Theorem 4.1, we showed that $\mu(n) = n-1$ for $n \geq 2$. There, a graph showing $\mu(n) \geq n-1$ is based on a disconnected graph. Therefore, we can investigate the maximum possible value for $\mu(G)$ when G is k-connected for $k \geq 1$. This problem remains open.

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