Matroid Representation of Clique Complexes

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Abstract. In this paper, we approach the quality of a greedy algorithm for the maximum weighted clique problem from the viewpoint of matroid theory. More precisely, we consider the clique complex of a graph (the collection of all cliques of the graph) and investigate the minimum number k such that the clique complex of a given graph can be represented as the intersection of k matroids. This number k can be regarded as a measure of "how complex a graph is with respect to the maximum weighted clique problem" since a greedy algorithm is a k-approximation algorithm for this problem. We characterize graphs whose clique complexes can be represented as the intersection of k matroids for any $k > 0$. Moreover, we determine the minimum number of matroids which we need to represent all graphs with n vertices. This number turns out to be exactly $n - 1$. Other related investigations are also given.

1 Introduction

A lot of combinatorial optimization problems can be seen as optimization problems on the corresponding independence systems. For example, for the minimum cost spanning tree problem the corresponding independence system is the collection of all forests of a given graph; for the maximum weighted matching problem the corresponding independence system is the collection of all matchings of a given graph; for the maximum weighted clique problem the corresponding independence system is the collection of all cliques of a given graph, which is called the clique complex of the graph. More examples are provided by Korte–Vygen [9].

It is known that any independence system can be represented as the intersection of some matroids. Jenkyns [7] and Korte–Hausmann [8] showed that a greedy algorithm is a k-approximation algorithm for the maximum weighted base problem of an independence system which can be represented as the intersection of k matroids. (Their result can be seen as a generalization of the validity of the

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greedy algorithm for matroids, shown by Rado [13] and Edmonds [3].) So the minimum number of matroids which we need to represent an independence system as their intersection is one of the measures of "how complex an independence system is with respect to the corresponding optimization problem."

In this paper, we investigate how many matroids we need to represent the clique complex of a graph as their intersection, while Fekete-Firla-Spille [5] investigated the same problem for matching complexes. We will show that the clique complex of a given graph G is the intersection of k matroids if and only if there exists a family of k stable-set partitions of G such that every edge of $\overline{G}$ is contained in a stable set of some stable-set partition in the family. This theorem implies that the problem to determine the clique complex of a given graph have a representation by k matroids or not belongs to NP (for any fixed k). This is not a trivial fact since in general the size of an independence system will be exponential when we treat it computationally.

The organization of this paper is as follows. In Sect. 2, we will introduce some terminology on independence systems. The proof of the main theorem will be given in Sect. 3. In Sect. 4, we will consider an extremal problem related to our theorem. In Sect. 5, we will investigate the case of two matroids more thoroughly. This case is significantly important since the maximum weighted base problem can be solved exactly in polynomial time for the intersection of two matroids [6]. (Namely, in this case, the maximum weighted clique problem can be solved in polynomial time for any non-negative weight vector by Frank's algorithm [6].) From the observation in that section, we can find the algorithm by Protti–Szwarcfiter [12] checks that a given clique complex has a representation by two matroids or not in polynomial time. We will conclude with Sect. 6.

2 Preliminaries

We will assume the basic concepts in graph theory. If you find something unfamiliar, see a textbook of graph theory (Diestel's book [2] or so). Here we will fix our notations. In this paper, all graphs are finite and simple unless stated otherwise. For a graph $G = (V, E)$ we denote the subgraph induced by $V' \subseteq V$ by $G[V']$. The complement of G is denoted by $\overline{G}$. The vertex set and the edge set of a graph $G = (V, E)$ are denoted by $V(G)$ and $E(G)$, respectively. A complete graph and a cycle with n vertices are denoted by K_n and C_n, respectively. The maximum degree, the chromatic number and the edge-chromatic number (or the chromatic index) of a graph G are denoted by $\Delta(G)$, $\chi(G)$ and $\chi'(G)$, respectively. A *clique* of a graph $G = (V, E)$ is a subset $C \subseteq V$ such that the induced subgraph $G[C]$ is complete. A *stable set* of a graph $G = (V, E)$ is a subset $S \subseteq V$ such that the induced subgraph $G[S]$ has no edge.

Now we introduce some notions of independence systems and matroids. For details of them, see Oxley's book [11]. Given a non-empty finite set V, an *independence system* on V is a non-empty family $\mathcal{I}$ of subsets of V satisfying: $X \in \mathcal{I}$ implies $Y \in \mathcal{I}$ for all $Y \subseteq X \subseteq V$. The set V is called the *ground set* of this independence system. In the literature, an independence system is also called an *abstract simplicial complex*. A *matroid* is an independence system $\mathcal{I}$ additionally satisfying the following *augmentation axiom*: for $X, Y \in \mathcal{I}$ with $|X| > |Y|$ there exists $z \in X \setminus Y$ such that $Y \cup \{z\} \in \mathcal{I}$. For an independence system $\mathcal{I}$, a set $X \in \mathcal{I}$ is called *independent* and a set $X \notin \mathcal{I}$ is called *dependent*. A *base* and a *circuit* of an independence system is a maximal independent set and a minimal dependent set, respectively. We denote the family of bases of an independence system $\mathcal{I}$ and the family of circuits of $\mathcal{I}$ by $\mathcal{B}(\mathcal{I})$ and $\mathcal{C}(\mathcal{I})$, respectively. Note that we can reconstruct an independence system $\mathcal{I}$ from $\mathcal{B}(\mathcal{I})$ or $\mathcal{C}(\mathcal{I})$ as $\mathcal{I} = \{X \subseteq V : X \subseteq B \text{ for some } B \in \mathcal{B}(\mathcal{I})\}$ and $\mathcal{I} = \{X \subseteq V : C \not\subseteq X \text{ for all } C \in \mathcal{C}(\mathcal{I})\}$. In particular, $\mathcal{B}(\mathcal{I}_1) = \mathcal{B}(\mathcal{I}_2)$ if and only if $\mathcal{I}_1 = \mathcal{I}_2$; similarly $\mathcal{C}(\mathcal{I}_1) = \mathcal{C}(\mathcal{I}_2)$ if and only if $\mathcal{I}_1 = \mathcal{I}_2$. We can see that the bases of a matroid have the same size from the augmentation axiom, but it is not the case for a general independence system.

Let $\mathcal{I}$ be a matroid on V. An element $x \in V$ is called a *loop* if $\{x\}$ is a circuit of $\mathcal{I}$. We say that $x, y \in V$ are *parallel* if $\{x, y\}$ is a circuit of the matroid $\mathcal{I}$. The next is well known.

Lemma 2.1 (see [11]). *For a matroid without a loop, the relation that "x is parallel to y" is an equivalence relation.*

Let $\mathcal{I}_1, \mathcal{I}_2$ be independence systems on the same ground set V. The *intersection* of $\mathcal{I}_1$ and $\mathcal{I}_2$ is just $\mathcal{I}_1 \cap \mathcal{I}_2$. The intersection of more independence systems is defined in a similar way. Note that the intersection of independence systems is also an independence system. Also note that the family of circuits of $\mathcal{I}_1 \cap \mathcal{I}_2$ is the family of the minimal sets of $\mathcal{C}(\mathcal{I}_1) \cup \mathcal{C}(\mathcal{I}_2)$, i.e., $\mathcal{C}(\mathcal{I}_1 \cap \mathcal{I}_2) = \min(\mathcal{C}(\mathcal{I}_1) \cup \mathcal{C}(\mathcal{I}_2))$. The following well-known observation is crucial in this paper.

Lemma 2.2 (see [4, 5, 9]). *Every independence system can be represented as the intersection of finitely many matroids on the same ground set.*

Proof. Denote the circuits of an independence system $\mathcal{I}$ by $C^{(1)}, \ldots, C^{(m)}$, and consider the matroid $\mathcal{I}_i$ with a unique circuit $\mathcal{C}(\mathcal{I}_i) = \{C^{(i)}\}$ for each $i \in \{1, \ldots, m\}$. Then, the family of the circuits of $\bigcap_{i=1}^{m} \mathcal{I}_i$ is nothing but $\{C^{(1)}, \ldots, C^{(m)}\}$. Therefore, we have $\mathcal{I} = \bigcap_{i=1}^{m} \mathcal{I}_i$. □

Due to Lemma 2.2, we are interested in representation of an independence system as the intersection of matroids. From the construction in the proof of Lemma 2.2, we can see that the number of matroids which we need to represent an independence system $\mathcal{I}$ by the intersection is at most $|\mathcal{C}(\mathcal{I})|$. However, we might do better. In this paper, we investigate such a number for a clique complex.

3 Clique Complexes and the Main Theorem

A graph gives rise to various independence systems. Among them, we will investigate clique complexes.

The *clique complex* of a graph $G = (V, E)$ is the collection of all cliques of G. We denote the clique complex of G by $\mathfrak{C}(G)$. Note that the empty set is a clique and $\{v\}$ is also a clique for each $v \in V$. So we can see that the clique complex is actually an independence system on V. We also say that an independence system is a clique complex if it is isomorphic to the clique complex of some graph. Notice that a clique complex is also called a *flag complex* in the literature.

Here we give some subclasses of the clique complexes. (We omit necessary definitions.) (1) The family of the stable sets of a graph G is nothing but the clique complex of $\overline{G}$. (2) The family of the matchings of a graph G is the clique complex of the complement of the line graph of G, which is called the *matching complex* of G. (3) The family of the chains of a poset P is the clique complex of the comparability graph of P, which is called the *order complex* of P. (4) The family of the antichains of a poset P is the clique complex of the complement of the comparability graph of P.

The next lemma may be a folklore.

Lemma 3.1. *Let $\mathcal{I}$ be an independence system on a finite set V. Then, $\mathcal{I}$ is a clique complex if and only if the size of every circuit of $\mathcal{I}$ is two. In particular, the circuits of the clique complex of G are the edges of $\overline{G}$.*

Proof. Let $\mathcal{I}$ be the clique complex of $G = (V, E)$. Since a single vertex $v \in V$ forms a clique, the size of each circuit is greater than one. Each dependent set of size two is an edge of the complement. Observe that they are minimal dependent sets since the size of each dependent set is greater than one. Suppose that there exists a circuit C of size more than two. Then each two elements in C form an edge of G. Hence C is a clique. This is a contradiction.

Conversely, assume that the size of every circuit of $\mathcal{I}$ is two. Then construct a graph $G' = (V, E')$ with $E' = \{\{u, v\} \in \binom{V}{2} : \{u, v\} \notin \mathcal{C}(\mathcal{I})\}$. Consider the clique complex $\mathfrak{C}(G')$. By the opposite direction which we have just shown, we can see that $\mathcal{C}(\mathfrak{C}(G')) = \mathcal{C}(\mathcal{I})$. Therefore $\mathcal{I}$ is the clique complex of G'. □

Now we start studying the number of matroids which we need for the representation of a clique complex as their intersection. First we characterize the case in which we need only one matroid. (namely the case in which a clique complex is a matroid). To do this, we define a partition matroid. A *partition matroid* is a matroid $\mathcal{I}(\mathcal{P})$ associated with a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ of V defined as $\mathcal{I}(\mathcal{P}) = \{I \subseteq V : |I \cap P_i| \leq 1 \text{ for all } i \in \{1, \ldots, r\}\}$. Observe that $\mathcal{I}(\mathcal{P})$ is a clique complex. Indeed if we construct a graph $G_{\mathcal{P}} = (V, E)$ from $\mathcal{P}$ as $u, v \in V$ are adjacent in $G_{\mathcal{P}}$ if and only if u, v are elements of distinct partition classes in $\mathcal{P}$, then we can see that $\mathcal{I}(\mathcal{P}) = \mathfrak{C}(G_{\mathcal{P}})$. Note that $\mathcal{C}(\mathcal{I}(\mathcal{P})) = \{\{u, v\} \in \binom{V}{2} : \{u, v\} \subseteq P_i \text{ for some } i \in \{1, \ldots, r\}\}$. Also note that $G_{\mathcal{P}}$ is a complete r-partite graph with the partition $\mathcal{P}$. In the next lemma, the equivalence of (1) and (3) is also noticed by Okamoto [10].

Lemma 3.2. *Let G be a graph. Then the following are equivalent. (1) The clique complex of G is a matroid. (2) The clique complex of G is a partition matroid. (3) G is complete r-partite for some r.*

Proof. "(2) ⇒ (1)" is clear, and "(3) ⇒ (2)" is immediate from the discussion above. So we only have to show "(1) ⇒ (3)." Assume that the clique complex $\mathfrak{C}(G)$ is a matroid. By Lemma 3.1, every circuit of $\mathfrak{C}(G)$ is of size two, which corresponds to an edge of $\overline{G}$. So the elements of each circuit are parallel. Lemma 2.1 says that the parallel elements induce an equivalence relation on $V(G)$, which yields a partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ of $V(G)$. Thus, we can see that G is a complete r-partite graph with the vertex partition $\mathcal{P}$. □

For the case of more matroids, we use a stable-set partition. A *stable-set partition* of a graph $G = (V, E)$ is a partition $\mathcal{P} = \{P_1, \ldots, P_r\}$ of V such that each P_i is a stable set of G. The following theorem is the main result of this paper. It tells us how many matroids we need to represent a given clique complex.

Theorem 3.3. *Let $G = (V, E)$ be a graph. Then, the clique complex $\mathfrak{C}(G)$ can be represented as the intersection of k matroids if and only if there exist k stable-set partitions $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(k)}$ such that $\{u, v\} \in \binom{V}{2}$ is an edge of $\overline{G}$ if and only if $\{u, v\} \subseteq S$ for some $S \in \bigcup_{i=1}^{k} \mathcal{P}^{(i)}$ (in particular, $\mathfrak{C}(G) = \bigcap_{i=1}^{k} \mathcal{I}(\mathcal{P}^{(i)})$).*

To show the theorem, we use the following lemmas.

Lemma 3.4. *Let $G = (V, E)$ be a graph. If the clique complex $\mathfrak{C}(G)$ can be represented as the intersection of k matroids, then there exist k stable-set partitions $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(k)}$ such that $\mathfrak{C}(G) = \bigcap_{i=1}^{k} \mathcal{I}(\mathcal{P}^{(i)})$.*

Proof. Assume that $\mathfrak{C}(G)$ is represented as the intersection of k matroids $\mathcal{I}_1$, $\ldots$, $\mathcal{I}_k$. Choose $\mathcal{I}_i$ arbitrarily ($i \in \{1, \ldots, k\}$). Then the parallel elements of $\mathcal{I}_i$ induce an equivalence relation on V. Let $\mathcal{P}^{(i)}$ be the partition of V arising from this equivalence relation. Then the two-element circuits of $\mathcal{I}_i$ are the circuits of $\mathcal{I}(\mathcal{P}^{(i)})$. Moreover, there is no loop in $\mathcal{I}_i$ (otherwise $\bigcap \mathcal{I}_i$ cannot be a clique complex). Therefore, we have that $\min(\bigcup_{i=1}^{k} \mathcal{C}(\mathcal{I}_i)) = \min(\bigcup_{i=1}^{k} \mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)})))$, which means that $\mathfrak{C}(G) = \bigcap_{i=1}^{k} \mathcal{I}_i = \bigcap_{i=1}^{k} \mathcal{I}(\mathcal{P}^{(i)})$. □

Lemma 3.5. *Let $G = (V, E)$ be a graph and $\mathcal{P}$ be a partition of V. Then $\mathfrak{C}(G) \subseteq \mathcal{I}(\mathcal{P})$ if and only if $\mathcal{P}$ is a stable-set partition of G.*

Proof. Assume that $\mathcal{P}$ is a stable-set partition of G. Take $I \in \mathfrak{C}(G)$ arbitrarily. Then we have $|I \cap P| \leq 1$ for each $P \in \mathcal{P}$ by the definitions of cliques and stable sets. Hence $I \in \mathcal{I}(\mathcal{P})$, namely $\mathfrak{C}(G) \subseteq \mathcal{I}(\mathcal{P})$. Conversely, assume that $\mathfrak{C}(G) \subseteq \mathcal{I}(\mathcal{P})$. Take $P \in \mathcal{P}$ and a clique C of G arbitrarily. From our assumption, we have $C \in \mathcal{I}(\mathcal{P})$. Therefore, it holds that $|C \cap P| \leq 1$. This means that P is a stable set of G, namely $\mathcal{P}$ is a stable-set partition of G. □

Now it is time to prove Theorem 3.3.

Proof (of Theorem 3.3). Assume that a given clique complex $\mathfrak{C}(G)$ is represented as the intersection of k matroids $\mathcal{I}_1, \ldots, \mathcal{I}_k$. From Lemma 3.4, $\mathfrak{C}(G)$ can be represented as the intersection of k matroids associated with stable-set partitions $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(k)}$ of G. We will show that these partitions $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(k)}$ satisfy the condition in the statement of the theorem. By Lemma 3.1, $\{u, v\}$ is an edge of $\overline{G}$ if and only if $\{u, v\}$ is a circuit of the clique complex $\mathfrak{C}(G)$, namely $\{u, v\} \in \mathcal{C}(\mathfrak{C}(G))$ $= \min(\bigcup_{i=1}^{k} \mathcal{C}(\mathcal{I}_i)) = \min(\bigcup_{i=1}^{k} \mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)}))) = \bigcup_{i=1}^{k} \mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)}))$. So this means that there exists at least one $i \in \{1, \ldots, k\}$ such that $\{u, v\} \in \mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)}))$. Hence, $\{u, v\} \subseteq S$ for some $S \in \mathcal{P}^{(i)}$ if and only if $\{u, v\}$ is an edge of $\overline{G}$.

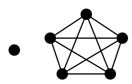
Conversely, assume that we are given a family of stable-set partitions $\mathcal{P}^{(1)}$, $\ldots$, $\mathcal{P}^{(k)}$ of V satisfying the condition in the statement of the theorem. We will show that $\mathfrak{C}(G) = \bigcap_{i=1}^k \mathcal{I}(\mathcal{P}^{(i)})$. By Lemma 3.5, we can see that $\mathfrak{C}(G) \subseteq \mathcal{I}(\mathcal{P}^{(i)})$ for all $i \in \{1, \ldots, k\}$. This shows that $\mathfrak{C}(G) \subseteq \bigcap_{i=1}^k \mathcal{I}(\mathcal{P}^{(i)})$. In order to show that $\mathfrak{C}(G) \supseteq \bigcap_{i=1}^k \mathcal{I}(\mathcal{P}^{(i)})$, we only have to show that $\mathcal{C}(\mathfrak{C}(G)) \subseteq \bigcup_{i=1}^k \mathcal{C}(\mathcal{I}(\mathcal{P}^{(I)}))$. Take $C \in \mathcal{C}(\mathfrak{C}(G))$ arbitrarily. By Lemma 3.1 we have $|C| = 2$. Set $C = \{u, v\} \in E(\overline{G})$. From our assumption, it follows that $\{u, v\} \subseteq S$ for some $S \in \bigcup_{i=1}^k \mathcal{P}^{(i)}$. This means that $\{u, v\} \in \bigcup_{i=1}^k \mathcal{C}(\mathcal{I}(\mathcal{P}^{(i)}))$. □

4 An Extremal Problem for a Clique Complex

Let $\mu(G)$ be the minimum number of matroids which we need for the representation of the clique complex of G as their intersection, and $\mu(n)$ be the maximum of $\mu(G)$ over all graphs G with n vertices. Namely, $\mu(G) = \min\{k : \mathfrak{C}(G) = \bigcap_{i=1}^k \mathcal{I}_i$ where $\mathcal{I}_1, \ldots, \mathcal{I}_k$ are matroids$\}$, and $\mu(n) = \max\{\mu(G) : G$ has n vertices$\}$. In this section, we will determine $\mu(n)$. From Lemma 2.2 we can immediately obtain $\mu(n) \leq \binom{n}{2}$. However, the following theorem tells us this is far from the truth.

Theorem 4.1. *For every $n \geq 2$, it holds that $\mu(n) = n - 1$.*

First we will prove that $\mu(n) \geq n - 1$. Consider the graph $K_1 \cup K_{n-1}$.



$K_1 \cup K_5$

Lemma 4.2. *For $n \geq 2$, we have $\mu(K_1 \cup K_{n-1}) = n - 1$, particularly $\mu(n) \geq n - 1$.*

Proof. $\overline{K_1 \cup K_{n-1}}$ has $n-1$ edges. From Lemma 3.1, the number of the circuits of $\mathfrak{C}(K_1 \cup K_{n-1})$ is $n-1$. By the argument below the proof of Lemma 2.2, we have $\mu(K_1 \cup K_{n-1}) \leq n-1$. Now, suppose that $\mu(K_1 \cup K_{n-1}) \leq n-2$. By Theorem 3.3 and the pigeon hole principle, there exists a stable-set partition $\mathcal{P}$ of $K_1 \cup K_{n-1}$ such that some class P in $\mathcal{P}$ contains at least two edges of $\overline{K_1 \cup K_{n-1}}$. However, this is impossible since P is stable. Hence, we have $\mu(K_1 \cup K_{n-1}) = n - 1$. □

Next we will prove that $\mu(n) \leq n - 1$. To do that, first we will look at the relationship of $\mu(G)$ with the edge-chromatic number.

Lemma 4.3. *We have $\mu(G) \leq \chi'(\overline{G})$ for any graph G with n vertices. Particularly, if n is even we have $\mu(G) \leq n - 1$ and if n is odd we have $\mu(G) \leq n$. Moreover, if $\mu(G) = n$ then n is odd and the maximum degree of $\overline{G}$ is $n-1$ (i.e., G has an isolated vertex).*

Proof. Consider a minimum edge-coloring of $\overline{G}$, and we will construct $\chi'(\overline{G})$ stable-set partitions of a graph G with n vertices from this edge-coloring.

We have the color classes $C^{(1)}, \dots, C^{(k)}$ of the edges where $k = \chi'(\overline{G})$. Let us take a color class $C^{(i)} = \{e_1^{(i)}, \dots, e_{l_i}^{(i)}\}$ ($i \in \{1, \dots, k\}$) and construct a stable-set partition $\mathcal{P}^{(i)}$ of G from $C^{(i)}$ as follows: S is a member of $\mathcal{P}^{(i)}$ if and only if either (1) S is a two-element set belonging to $C^{(i)}$ (i.e., $S = e_j^{(i)}$ for some $j \in \{1, \dots, l_i\}$) or (2) S is a one-element set $\{v\}$ which is not used in $C^{(i)}$ (i.e., $v \notin e_j^{(i)}$ for all $j \in \{1, \dots, l_i\}$). Notice that $\mathcal{P}^{(i)}$ is actually a stable-set partition. Then we collect all the stable-set partitions $\mathcal{P}^{(1)}, \dots, \mathcal{P}^{(k)}$ constructed by the procedure above. Moreover, we can check that these stable-set partitions satisfy the condition in Theorem 3.3. Hence, we have $\mu(G) \leq k = \chi'(\overline{G})$.

Here, notice that $\chi'(\overline{G}) \leq \chi'(K_n)$. So if n is even, then $\chi'(K_n)$ is $n-1$, which concludes $\mu(G) \leq n-1$. If n is odd, then $\chi'(K_n)$ is n, which concludes $\mu(G) \leq n$.

Assume that $\mu(G) = n$. From the discussion above, n should be odd. Remark that Vizing's theorem says for a graph H with maximum degree $\Delta(H)$ we have $\chi'(H) = \Delta(H)$ or $\Delta(H)+1$. If $\Delta(\overline{G}) \leq n-1$, then we have that $\mu(G) \leq \chi'(\overline{G}) \leq \Delta(\overline{G}) + 1 \leq n$. So $\mu(G) = n$ holds only if $\Delta(\overline{G}) + 1 = n$. □

Next we will show that if a graph G with an odd number of vertices has an isolated vertex then $\mu(G) \leq n - 1$. This completes the proof of Theorem 4.1.

Lemma 4.4. *Let n be odd and G be a graph with n vertices which has an isolated vertex. Then $\mu(G) \leq n - 1$.*

Proof. Let v^* be an isolated vertex of G. Consider the subgraph of G induced by $V(G) \setminus \{v^*\}$. Call this induced subgraph G'. Since G' has $n - 1$ vertices, which is even, we have $\mu(G') \leq n - 2$ from Lemma 4.3.

Now we will construct $n-1$ stable-set partitions of G which satisfy the condition in Theorem 3.3 from $n - 2$ stable-set partitions of G' which also satisfy the condition in Theorem 3.3. Denote the vertices of G' by $1, \dots, n - 1$, and the stable-set partitions of G' by $\mathcal{P'}^{(1)}, \dots, \mathcal{P'}^{(n-2)}$. Then construct stable-set partitions $\mathcal{P}^{(1)}, \dots, \mathcal{P}^{(n-2)}, \mathcal{P}^{(n-1)}$ of G as follows. For $i = 1, \dots, n - 2$, $S \in \mathcal{P}^{(i)}$ if and only if either (1) $S \in \mathcal{P'}^{(i)}$ and $i \notin S$ or (2) $v^* \in S$, $S \setminus \{v^*\} \in \mathcal{P'}^{(i)}$ and $i \in S$. Also $S \in \mathcal{P}^{(n-1)}$ if and only if either (1) $S = \{v^*, n - 1\}$ or (2) $S = \{i\}$ ($i = 1, \dots, n-2$). We can observe that the stable-set partitions $\mathcal{P}^{(1)}, \dots, \mathcal{P}^{(n-1)}$ satisfy the condition in Theorem 3.3 since v^* is an isolated vertex of G. □

5 Characterization for Two Matroids

In this section, we will look more closely at a clique complex which can be represented as the intersection of two matroids. Note that Fekete–Firla–Spille [5] gave a characterization of the graphs whose matching complexes can be represented as the intersections of two matroids. So the theorem in this section is a generalization of their result.

To do this, we introduce another concept. The *stable-set graph* of a graph $G = (V, E)$ is a graph whose vertices are the maximal stable sets of G and two vertices of which are adjacent if the corresponding two maximal stable sets share a vertex in G. We denote the stable-set graph of a graph G by $\mathcal{S}(G)$.

Lemma 5.1. *Let G be a graph. Then the clique complex $\mathfrak{C}(G)$ can be represented as the intersection of k matroids if the stable-set graph $\mathcal{S}(G)$ is k-colorable.*

Proof. Assume that we are given a k-coloring c of $\mathcal{S}(G)$. Then gather the maximal stable sets of G which have the same color with respect to c, that is, put $C_i = \{S \in V(\mathcal{S}(G)) : c(S) = i\}$ for all $i = 1, \dots, k$. We can see that the members of C_i are disjoint maximal stable sets of G for each i.

Now we construct a graph G_i from C_i as follows. The vertex set of G_i is the same as that of G, and two vertices of G_i are adjacent if and only if either (1) one belongs to a maximal stable set in C_i and the other belongs to another maximal stable set in C_i, or (2) one belongs to a maximal stable set in C_i and the other belongs to no maximal stable set in C_i. Remark that G_i is complete r-partite, where r is equal to $|C_i|$ plus the number of the vertices which do not belong to any maximal stable set in C_i. Then consider $\mathfrak{C}(G_i)$, the clique complex of G_i. By Lemma 3.2, we can see that $\mathfrak{C}(G_i)$ is actually a matroid. Since an edge of G is also an edge of G_i, we have that $\mathfrak{C}(G) \subseteq \mathfrak{C}(G_i)$.

Here we consider the intersection $\mathcal{I} = \bigcap_{i=1}^{k} \mathfrak{C}(G_i)$. Since $\mathfrak{C}(G) \subseteq \mathfrak{C}(G_i)$ for any i, we have $\mathfrak{C}(G) \subseteq \mathcal{I}$. Since each circuit of $\mathfrak{C}(G)$ is also a circuit of $\mathfrak{C}(G_i)$ for some i (recall Lemma 3.1), we also have $\mathcal{C}(\mathfrak{C}(G)) \subseteq \mathcal{C}(\mathcal{I})$, which implies $\mathfrak{C}(G) \supseteq \mathcal{I}$. Thus we have $\mathfrak{C}(G) = \mathcal{I}$. □

Note that the converse of Lemma 5.1 does not hold even if $k = 3$. A counterexample is the graph $G = (V, E)$ defined as $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$. Here $\mathfrak{C}(G)$ is represented as the intersection of three matroids $\mathcal{C}(\mathcal{I}_1) = \{\{1, 3, 5\}, \{2, 4, 6\}\}$, $\mathcal{C}(\mathcal{I}_2) = \{\{1, 3, 6\}, \{2, 4, 5\}\}$ and $\mathcal{C}(\mathcal{I}_3) = \{\{1, 4, 5\}, \{2, 3, 6\}\}$ while $\mathcal{S}(G)$ is not 3-colorable but 4-colorable.

However, the converse holds if $k = 2$.

Theorem 5.2. *Let G be a graph. The clique complex $\mathfrak{C}(G)$ can be represented as the intersection of two matroids if and only if the stable-set graph $\mathcal{S}(G)$ is 2-colorable (i.e., bipartite).*

Proof. The if-part is straightforward from Lemma 5.1. We will show the only-if-part. Assume that $\mathfrak{C}(G)$ is represented as the intersection of two matroids. Thanks to Theorem 3.3, we assume that these two matroids are associated with stable-set partitions $\mathcal{P}^{(1)}, \mathcal{P}^{(2)}$ of G satisfying the condition in Theorem 3.3.

Let S be a maximal stable set of G. Now we will see that $S \in \mathcal{P}^{(1)} \cup \mathcal{P}^{(2)}$. From the maximality of S, we only have to show that $S \subseteq P$ for some $P \in \mathcal{P}^{(1)} \cup \mathcal{P}^{(2)}$. (Then, the maximality of S will tell us that $S = P$.) This claim clearly holds if $|S| = 1$. If $|S| = 2$, the claim holds from the condition in Theorem 3.3.

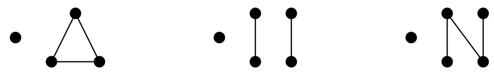
Assume that $|S| \geq 3$. Consider the following independence system $\mathcal{I} = \{I \subseteq S : I \subseteq P$ for some $P \in \mathcal{P}^{(1)} \cup \mathcal{P}^{(2)}\}$. Take a base B of $\mathcal{I}$ arbitrarily. Since $B \subseteq S$,

B is a dependent set of $\mathfrak{C}(G)$. So B contains a circuit of $\mathfrak{C}(G)$. By Lemma 3.1, we have $|B| \geq 2$. Suppose that $S \setminus B \neq \emptyset$ for a contradiction. Pick up $u \in S \setminus B$. Assume that $B \subseteq P$ for some $P \in \mathcal{P}^{(1)}$, without loss of generality. Then $\{u, v\}$ is a circuit of $\mathfrak{C}(G)$ for any $v \in B$ since S is a stable set of G. Moreover, u and v belong to different sets of $\mathcal{P}^{(1)}$ (otherwise, it would violate the maximality of B). From the condition in Theorem 3.3, there should exist some $P' \in \mathcal{P}^{(2)}$ such that $\{u, v\} \subseteq P'$ for all $v \in B$. By the transitivity of the equivalence relation induced by $\mathcal{P}^{(2)}$, we have $\{u\} \cup B \subseteq P'$. This contradicts the maximality of B. Therefore, we have $S = B$, which means that $S \in \mathcal{P}^{(1)} \cup \mathcal{P}^{(2)}$.

Now we color the vertices of $\mathcal{S}(G)$, i.e., the maximal stable sets of G. If a maximal stable set S belongs to $\mathcal{P}^{(1)}$, then S is colored by 1. Similarly if S belongs to $\mathcal{P}^{(2)}$, then S is colored by 2. (If S belongs to both, then S is colored by either 1 or 2 arbitrarily.) This coloring certainly provides a proper 2-coloring of $\mathcal{S}(G)$ since $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ are stable-set partitions of G. □

Some researchers already noticed that the bipartiteness of $\mathcal{S}(G)$ is characterized by other properties. We gather them in the following proposition.

Proposition 5.3. *Let G be a graph. Then the following are equivalent. (1) The stable-set graph $\mathcal{S}(G)$ is bipartite. (2) G is the complement of the line graph of a bipartite multigraph. (3) $\overline{G}$ has no induced subgraph isomorphic to $K_1 \cup K_3$, $K_1 \cup K_2 \cup K_2$, $K_1 \cup P_3$ or $\overline{C_{2k+3}}$ $(k = 1, 2, \ldots)$.*



$K_1 \cup K_3$ $K_1 \cup K_2 \cup K_2$ $K_1 \cup P_3$

Proof. "(1) ⇔ (2)" is immediate from a result by Cai–Corneil–Proskurowski [1]. Also "(1) ⇔ (3)" is immediate from a result by Protti–Szwarcfiter [12]. □

From the view of Proposition 5.3, we can decide whether the stable-set graph of a graph is bipartite or not in polynomial time using the algorithm described by Protti–Szwarcfiter [12].

6 Concluding Remarks

In this paper, motivated by the quality of a natural greedy algorithm for the maximum weighted clique problem, we characterized the number k such that the clique complex of a graph can be represented as the intersection of k matroids (Theorem 3.3). This implies that the problem to determine the clique complex of a given graph has a representation by k matroids or not belongs to NP. Also, in Sect. 5 we observed that the corresponding problem for two matroids can be solved in polynomial time. However, the problem for three or more matroids is not known to be solved in polynomial time. We leave the further issue on computational complexity of this problem as an open problem.

Moreover, we showed that $n-1$ is exactly the maximum number of matroids we need for the representation of the clique complex of a graph with n vertices

(Theorem 4.1). This implies that the approximation ratio of the greedy algorithm for the maximum weighted clique problems is at most $n - 1$.

Furthermore, we can show the following theorem. In this theorem, we see a graph itself as an independence system: namely a subset of the vertex set is independent if and only if it is (1) the empty set, (2) a one-element set, or (3) a two-element set which forms an edge of the graph. A proof is omitted due to the page limitation.

Theorem 6.1. *A graph G can be represented as the intersection of k matroids if and only if the clique complex $\mathfrak{C}(G)$ can be represented as the intersection of k matroids.*

In this paper, we approached the quality of a greedy algorithm for the maximum weighted clique problem from the viewpoint of matroid theory. This approach might be useful for other combinatorial optimization problems.

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