Using Mixed-Integer-Programming
on the Order-Degree-Problem

GraphGolf Workshop (24.11.2021)

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Modelling the ODP (n, d) as a MIP (with ASPL)

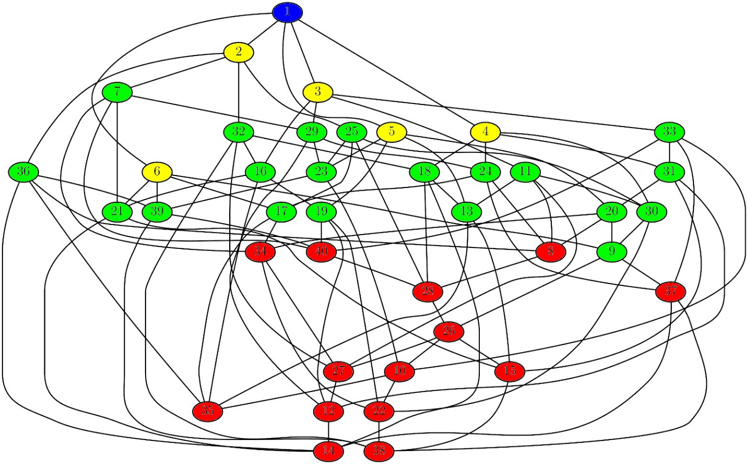
objective function	$\min \; 10 \cdot k + l$
diameter k	$\forall\, s, t \in V, s \neq t : SP_{st} \leq k$
ASPL l	$\frac{1}{n \cdot (n-1)} \sum_{s \in V} \sum_{\substack{t \in V \\ s \neq t}} SP_{st} = l$
APSP	$\forall\, s, t \in V, s \neq t : SP_{st} = ???$
degree d	$\forall\, i \in V : \sum_{\substack{j \in V \\ i \neq j}} z_{ij} \leq d$
	$\forall\, i, j \in V, i \neq j : z_{ij} \in \{0, 1\}$
	$\forall\, s, t \in V, s \neq t : SP_{st} \in \mathbb{N}$

APSP-Variations and their model size

- Classic Multi-Commodity-Flow: $O(n^4)$
- Quadratic Seidel-APSP: $O(n^2)$
- Linearized Seidel-APSP: $O(n^3)$

1. MCF-APSP model for competition instance (40, 5) exceeded 64GB memory limit of used test system
2. more/better established methods for linear models
3. limit search space by setting bounds (known or heuristic)
4. further tuning options by limiting the diameter k

Analysis of Optimal Solution for $(40, 5)$

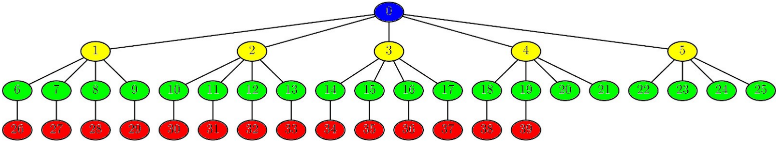


Assume (Heuristic Solution) Structure

less variables leads to faster (and most likely better) results:

- fix variables of inner tree structure (blue, yellow, green)
- connect red nodes with green nodes

reduce search space by problem-based symmetry-breaking



ToDo: look out for cutting off optimal solutions

Random, Greedy, Optimize

1. fast heuristic: connect green & red nodes randomly
2. slow heuristic: connect green & red nodes greedily

   both: link nodes with submaximal degree
   fast: add random edge, if feasible
   slow: choose longest path from possible pairs

3. assist optimization model:
   - reduce model size with fixings
   - start with good heuristic solutions

solutions always stay feasbile w.r.t. ODP

Calculation Results

focus: enumeration of all instances < 100 nodes
optimization model implemented with ZIMPL
generated MIP model files solved with Gurobi

- 4465 instances (excluding trivial by $3 \leq d \leq n - 3$)
- 2490 solved by heuristics (usually $d \geq \frac{n}{2}$ – ”easy”)
- 3574 solved by MIP models (”medium”)

observation on ”hard” instances

- odd $n \cdot d$
- $d < \frac{n}{4}$

usually due to unreachable lower bound

possible next steps

- combine structure assumption with known and future methods
- feasibility proof for tree structure (without edges to red nodes)
  - follow-up: further fixings for less symmetry?
  - follow-up: improvement of lower bounds for ”hard” instances?
- avoid MIP numeric issues for larger node counts

Links

- `www.zib.de/projects/research-campus-modal`
- modelling: `https://zimpl.zib.de`
- solver: `https://www.gurobi.com`
- framework: `https://www.scipopt.org`

quadratic Seidel-APSP

$$\forall\, s, t \in V :$$

$$SP_{st} = 1 + \sum_{j=1}^{n} (1 - dist_{stj})$$

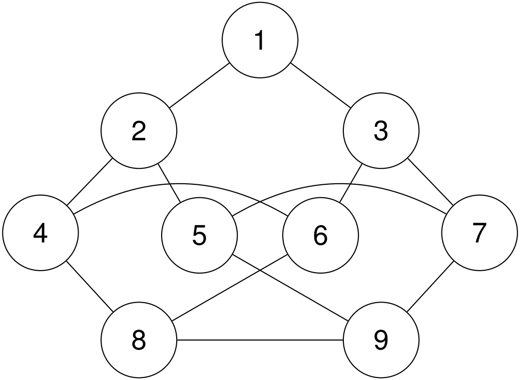
$$\forall\, j \in \{1, \dots, n-1\} : \forall\, s, t \in V :$$

$$dist_{st(j+1)} \leq dist_{stj} + \sum_{\substack{u \in V \\ s \neq u \neq t}} dist_{suj} \cdot dist_{ut1}$$

$$\forall\, s, t \in V, s \neq t : \forall\, j \in \{1, \dots, n\}$$

$$dist_{stj} \in \{0, 1\}$$

Optimal Solution of $(9, 3)$



$\rightsquigarrow$ very raw idea: construct tree-based structed solution by alternating links inside and between lower tree levels