Finding a Generalized Moore Graph Using Randomized Depth-First Search

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Overview

- In participating in Graph Golf, we focused on the deepest improvement award.
- Generalized Moore graphs (GMGs) are known to have the smallest average shortest path length (ASPL) among all d–regular graphs with order n [1], [2].
- We have developed some efficient depth-first search algorithms for finding GMGs[3].
- Despite these efforts, the algorithm still suffers from backtrackings.
- We show that the computations can be reduced by randomly traversing the search trees.
- We found GMGs that were not found using the deterministic search.

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[1] V. G. Cerf *et al.*, *Congressus Numerantium*, vol. 9, pp. 379–398, 1973.

[2] V. G. Cerf *et al.*, *Networks*, vol. 4, no. 4, pp. 335–342, 1974.

[3] Y. Satotani *et al.*, *TENCON 2018 - 2018 IEEE Region 10 Conference*, pp. 0832–0837, 2018.

Generalized Moore Graph

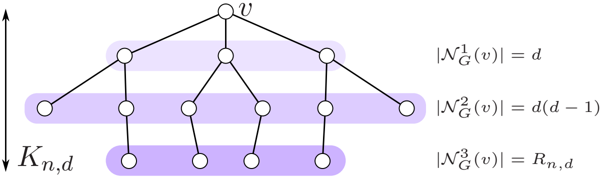
Definition (Cerf et al.[1])

A d-regular graph G with order n is called a generalized Moore graph (GMG) if the following holds for all vertices v.

$$|\mathcal{N}_G^l(v)| = \begin{cases} d(d-1)^{l-1}, & \text{if } 1 \le l < K_{n,d}, \\ R_{n,d}, & \text{if } l = K_{n,d}, \\ 0, & \text{if } K_{n,d} < l \end{cases}$$

$$K_{n,d} = \max\left\{ k \;\middle|\; n - 1 - \sum_{l=1}^{k} d(d-1)^{l-1} \ge 0 \right\} + 1, \quad R_{n,d} = n - 1 - \sum_{l=1}^{K_{n,d}-1} d(d-1)^{l-1}.$$

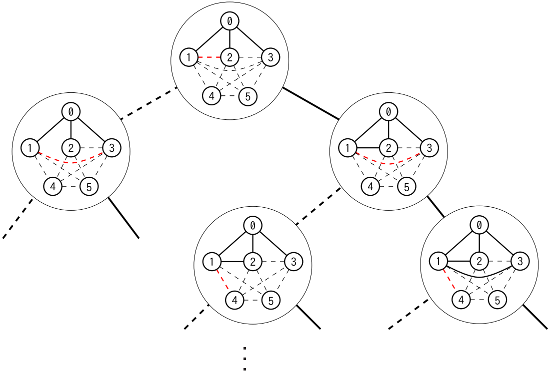
A spanning trees of this structure exists for each vertex v in a GMG.



[1] V. G. Cerf *et al.*, *Congressus Numerantium*, vol. 9, pp. 379–398, 1973.

Depth-First Search Algorithm

- Each node in the search tree contains a graph and a set of edges to include/exclude.



- To check if a node contains a GMG, we may need to create a regular graph.
- Is it possible to know if a node will reach the GMG before creating a regular graph?

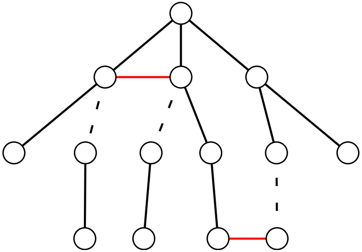
Property of GMGs

Theorem (Satotani and Takahashi[3])

A d-regular graph G with order n is a GMG if and only if both of the following two conditions are satisfied.

1. G has no cycle with length less than $2K_{n,d} - 1$.
2. The diameter of G is $K_{n,d} - 1$ if $R_{n,d} = 0$, and $K_{n,d}$ if $R_{n,d} > 0$.

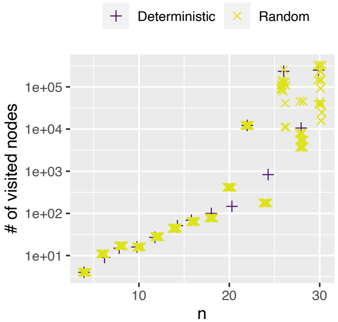
- If any cycle of length less than $2K_{n,d} - 1$ will be made including the edge, exclude it.
- If the edge is excluded and the diameter exceed $K_{n,d}$ despite including all remaining edges, include the edge.



[3] Y. Satotani *et al.*, *TENCON 2018 - 2018 IEEE Region 10 Conference*, pp. 0832–0837, 2018.

Proposed Method and Performance Evaluation

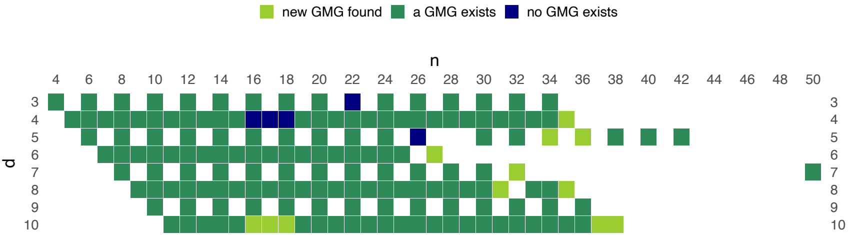
- Our previous method is deterministic (always "include" then "exclude").
  - This helps the graph grow faster at the early stage, but backtrackings can occur at the later stage (no feasible next node).
- We propose a new algorithm that randomly traverse the search tree in an attempt to reduce the backtrackings.
- An experiment to compare their performance
  - evaluation metric: the number of nodes the algorithms visited
  - random traversal algorithm runs 20 time
  - $n \in \{4, 6, \ldots, 30\}$, $d = 3$



- For $n = 24, 26, 28, 30$, the random algorithm visits fewer nodes for many runs.
- For $n = 22$, the number of visited nodes are the same because there is no GMG[4].
- For $n = 20$, the random algorithm visits more nodes.

[4] A. J. Hoffman *et al.*, *IBM Journal of Research and Development*, vol. 4, no. 5, pp. 497–504, 1960.

Existence of a GMG and Summary



- Contributions
  - We proposed a new method for finding a GMG that randomly traverses the search trees to reduce backtrackings.
  - We confirmed through experiments that the random traversal reduces the computational costs in some settings.
  - We found 12 new GMGs.
- Future work
  - Developing a cost function of a node for more efficient algorithm (such as A^*).

Reference

[1] V. G. Cerf, D. D. Cowan, R. C. Mullin, and R. G. Stanton, "Computer networks and generalized Moore graphs," *Congressus Numerantium*, vol. 9, pp. 379–398, 1973.

[2] V. G. Cerf, D. D. Cowan, R. C. Mullin, and R. G. Stanton, "A lower bound on the average shortest path length in regular graphs," *Networks*, vol. 4, no. 4, pp. 335–342, 1974.

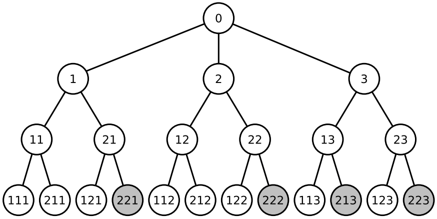
[3] Y. Satotani and N. Takahashi, "Depth-first search algorithms for finding a generalized moore graph," in *TENCON 2018 - 2018 IEEE Region 10 Conference*, 2018, pp. 0832–0837.

[4] A. J. Hoffman and R. R. Singleton, "On Moore graphs with diameters 2 and 3," *IBM Journal of Research and Development*, vol. 4, no. 5, pp. 497–504, 1960.

Appendix: The Root Node of a Search Tree

We used following spanning tree as the root node in search trees[3].

1. Construct a balanced tree with depth $K_{n,d}$ and degree d.
2. Label each node as follows:
   - root node is 0,
   - children of root are $1, \ldots, d$,
   - i-th child node of a node labeled in $l_m l_{m-1} \ldots l_1$ is $i l_m l_{m-1} \ldots l_1$.
3. Delete $d(d-1)^{K_{n,d}} - R_{n,d}$ nodes with the highest label values.

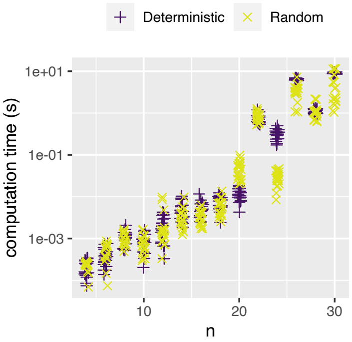


However, the validity of using these spanning trees is unknown.

[3] Y. Satotani *et al.*, *TENCON 2018 - 2018 IEEE Region 10 Conference*, pp. 0832–0837, 2018.

Appendix: Comparison of Computation Time

- We also compared the computation time.



- We can see the characteristics similar to those seen in the comparison of visited nodes.