Abstract Families of Abstract Categorial Languages

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ACGs and AFLs

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- Suggests machine models for ACGs.

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A family of languages is an AFL if it is closed under

- union ($\cup$), concatenation ($\cdot$), positive closure ($^+$);
- ϵ-free homomorphism (ϵ-free h);
- inverse homomorphism (h^{-1});
- intersection with regular sets ($\cap R$)

Examples of (full) AFLs

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PTIME is not an AFL unless $\mathrm{P} = \mathrm{NP}$.

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Fact.
A family of languages is closed under h, h^{-1}, $\cap R$ iff it is closed under finite transductions.

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Fact.
A family of languages is closed under h, h^{-1}, $\cap R$ iff it is closed under finite transductions.

Theorem (Ginsburg and Greibach 1969).
Full AFLs are exactly characterized by abstract families of acceptors.

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We prove closure under h, h^{-1}, $\cap R$, using some technical properties of the Curry-style type assignment system $\lambda\!\!\rightarrow$.

Type assignment system $\lambda{\to}_\Sigma$

$\Sigma = \langle A, C, \tau \rangle$: higher-order signature

Write $M, N, P, \ldots$ for λ-terms.

$$\vdash_\Sigma c : \tau(c) \qquad x : \alpha \vdash_\Sigma x : \alpha$$

$$\frac{\Gamma, (x : \alpha)^\circ \vdash_\Sigma M : \beta}{\Gamma \vdash_\Sigma \lambda x.M : \alpha \to \beta} \qquad \frac{\Gamma \vdash_\Sigma M : \alpha \to \beta \quad \Delta \vdash_\Sigma N : \alpha}{\Gamma, \Delta \vdash_\Sigma MN : \beta}$$

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$\mathscr{L} = \langle \sigma, \theta \rangle$: lexicon from Σ_1 to Σ_2

$$\vdash_{\Sigma_2} \theta(c) : \sigma(\tau_1(c))$$

$\theta(c)$: a closed linear λ-term built upon Σ_2.

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Write $|M|_\beta$ for the β-normal form of M.

Properties of lexicons

β-reduction commutes with lexicons:

$$M \twoheadrightarrow_\beta M' \quad \text{implies} \quad \mathscr{L}(M) \twoheadrightarrow_\beta \mathscr{L}(M').$$

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$$\Gamma \vdash_{\Sigma_1} M : \alpha \quad \text{implies} \quad \mathscr{L}(\Gamma) \vdash_{\Sigma_2} \mathscr{L}(M) : \mathscr{L}(\alpha).$$

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If $\mathscr{L}_1 = \langle \sigma_1, \theta_1 \rangle$ is a lexicon from Σ_1 to Σ_2 and $\mathscr{L}_2 = \langle \sigma_2, \theta_2 \rangle$ is a lexicon from Σ_2 to Σ_3, then

$$\mathscr{L}_2 \circ \mathscr{L}_1 = \langle \sigma_2 \circ \sigma_1, \theta_2 \circ \theta_1 \rangle$$

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Important facts about $\lambda{\rightarrow}_{\Sigma}$

Subject Reduction Theorem.

If $\Gamma \vdash_{\Sigma} M : \alpha$ and $M \twoheadrightarrow_{\beta} M'$, then $\Gamma \vdash_{\Sigma} M' : \alpha$.

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If $\Gamma \vdash_{\Sigma} M' : \alpha$ and $M \twoheadrightarrow_{\beta} M'$ by non-erasing non-duplicating β-reduction, then $\Gamma \vdash_{\Sigma} M : \alpha$.

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If M is a λI-term and $\Gamma \vdash_\Sigma M : \alpha$, then there is a unique $\lambda{\to}_\Sigma$-deduction of this judgment.

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Principal Pair Theorem.
If $\Gamma \vdash M : \alpha$ then there is a most general such $\langle \Gamma, \alpha \rangle$ (called a principal pair for M).

ACGs for string languages

Let $\mathscr{G} = \langle \Sigma_1, \Sigma_2, \mathscr{L}, s \rangle$ where

$$\begin{aligned}
\Sigma_1 &= \langle A_1, C_1, \tau_1 \rangle, \\
\Sigma_2 &= \langle \{o\}, C_2, \tau_2 \rangle, \\
s &\in A_1, \\
\tau_2(a) &= o \to o \quad \text{for all } a \in C_2, \\
\mathscr{L} &= \langle \sigma, \theta \rangle, \\
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$o \to o$ is the type of string.

For $a_1, \ldots, a_n \in C_2$, $/a_1 \ldots a_n/$ stands for $\lambda x. a_1(\ldots(a_n x)\ldots)$.

Closure under h

Let $h\colon C_2^* \to C_3^*$ be a homomorphism, and define

$$\Sigma_3 = \langle \{o\}, C_3, \tau_3 \rangle,$$

$$\tau_3(b) = o \to o \quad \text{for all } b \in C_3,$$

$$\mathscr{L}_h = \langle \mathrm{id}, \theta_h \rangle \quad \text{lexicon from } \Sigma_2 \text{ to } \Sigma_3,$$

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$$\mathscr{G}_h = \langle \Sigma_1, \Sigma_3, \mathscr{L}_h \circ \mathscr{L}, s \rangle.$$

Then

$$\mathcal{O}(\mathscr{G}_h) = \{ /h(w)/ \mid /w/ \in \mathcal{O}(\mathscr{G}) \}.$$

Closure under $\cap R$

Let $M = \langle C_2, Q, \delta, q_I, \{q_F\}\rangle$ be an NFA without ϵ-transitions with just one final state.

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Define a signature $\Sigma_M = \langle Q, C_M, \tau_M \rangle$ by

$$C_M = \{\, a^{r \to q} \mid a \in C_2 \text{ and } r \in \delta(q, a) \,\},$$
$$\tau_M(a^{r \to q}) = r \to q \quad \text{for all } a^{r \to q} \in C_M.$$

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We have $\vdash_{\Sigma_M} N : q_F \to q_I$ iff $\mathscr{L}_2(N) =_{\beta\eta} /w/$ for some $w \in L(M)$.

Closure under $\cap R$ (continued)

Define another signature $\Sigma_{\cap R} = \langle A_{\cap R}, C_{\cap R}, \tau_{\cap R} \rangle$ by

$$A_{\cap R} = \{\, p^{\beta} \mid p \in A_1, \beta \in \mathscr{T}(Q), \mathscr{L}_2(\beta) = \mathscr{L}(p) \,\},$$

$$\begin{aligned} C_{\cap R} = \{\, d_{\langle c, N, \beta \rangle} \mid {} & c \in C_1, N \in \Lambda(\Sigma_M), \beta \in \mathscr{T}(Q), \\ & \vdash_{\Sigma_M} N : \beta, \mathscr{L}_2(N) = \mathscr{L}(c), \\ & \mathscr{L}_2(\beta) = \mathscr{L}(\tau_1(c)) \,\}, \end{aligned}$$

$$\tau_{\cap R}(d_{\langle c, N, \beta \rangle}) = \mathrm{anti}(\tau_1(c), \beta)$$

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where

$$\mathrm{anti}(\alpha_1 \to \alpha_2, \beta_1 \to \beta_2) = \mathrm{anti}(\alpha_1, \beta_1) \to \mathrm{anti}(\alpha_2, \beta_2)$$

$$\mathrm{anti}(p, \beta) = p^{\beta}$$

Closure under $\cap R$ (continued)

$\tau_{\cap R}(d_{\langle c,N,\beta\rangle}) = \text{anti}(\tau_1(c), \beta)$ is always defined and is a most specific common anti-instance of $\tau_1(c)$ and β.

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Define a lexicon $\mathscr{L}_1 = \langle \sigma_1, \theta_1 \rangle$ from $\Sigma_{\cap R}$ to Σ_1 and a lexicon $\mathscr{L}_M = \langle \sigma_M, \theta_M \rangle$ from $\Sigma_{\cap R}$ to Σ_M:

$$\begin{aligned}
\sigma_1(p^\beta) &= p \quad \text{for all } p^\beta \in A_{\cap R}, \\
\theta_1(d_{\langle c,N,\beta\rangle}) &= c \quad \text{for all } d_{\langle c,N,\beta\rangle} \in C_{\cap R}, \\
\sigma_M(p^\beta) &= \beta \quad \text{for all } p^\beta \in A_{\cap R}, \\
\theta_M(d_{\langle c,N,\beta\rangle}) &= N \quad \text{for all } d_{\langle c,N,\beta\rangle} \in C_{\cap R}.
\end{aligned}$$

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Define an ACG $\mathscr{G}_{\cap R} = \langle \Sigma_{\cap R}, \Sigma_2, s^{q_F \to q_I}, \mathscr{L}_{\cap R} \rangle$ by

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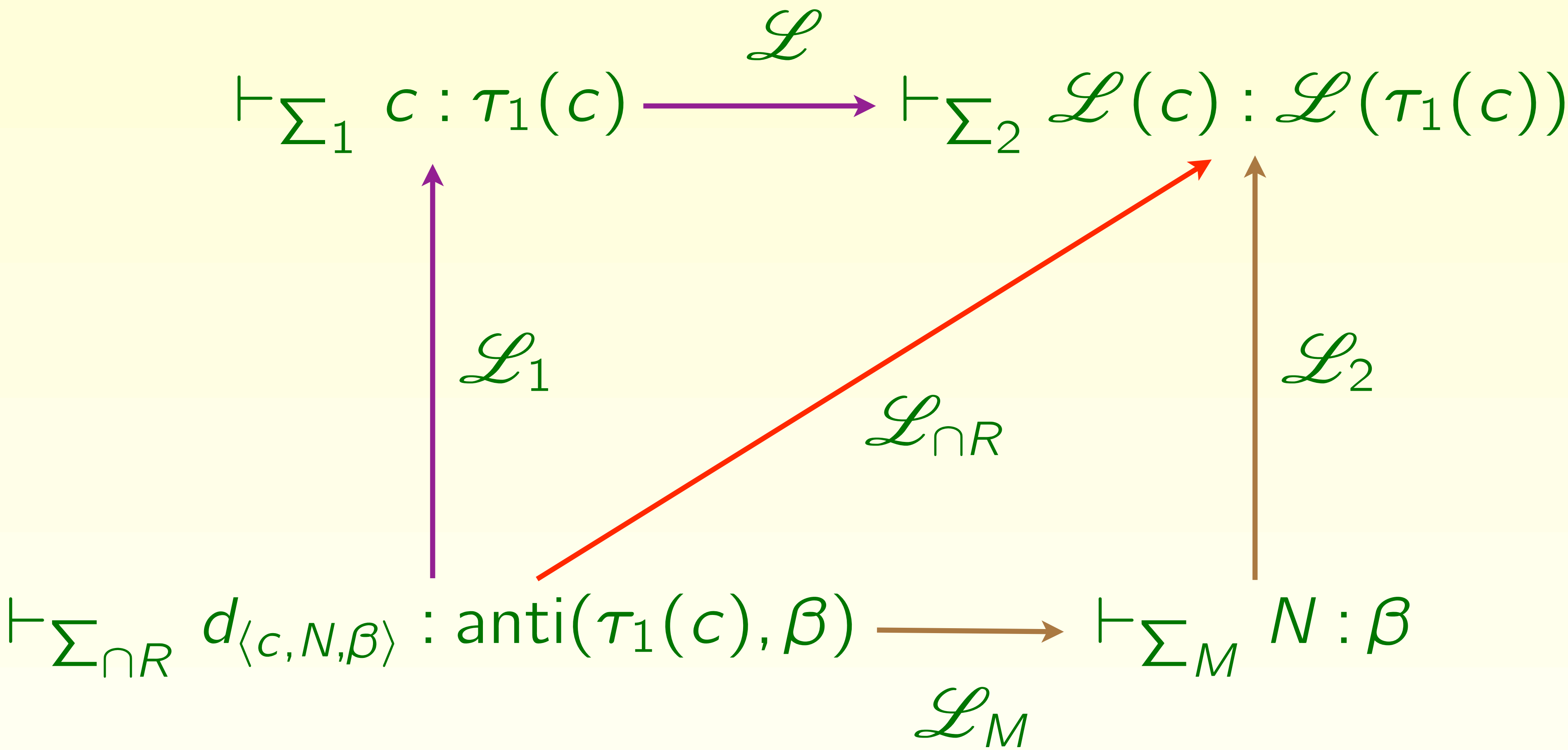
Lemma.

$$\mathscr{L}_{\cap R} = \mathscr{L} \circ \mathscr{L}_1, \qquad \mathscr{L}_{\cap R} = \mathscr{L}_2 \circ \mathscr{L}_M.$$

Closure under $\cap R$ (continued)



Closure under $\cap R$ (continued)

Lemma. $\mathcal{O}(\mathscr{G}_{\cap R}) \subseteq \mathcal{O}(\mathscr{G}) \cap \{\, /w/ \mid w \in L(M) \,\}$.

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Proof.

Suppose $/a_1 \dots a_n/ \in \mathscr{O}(\mathscr{G}_{\cap R})$. Let $P \in \mathcal{A}(\mathscr{G}_{\cap R})$ be such that $\mathscr{L}(P) \twoheadrightarrow_\beta /a_1 \dots a_n/$. Since

$$\vdash_{\Sigma_{\cap R}} P : s^{q_F \to q_I}, \tag{1}$$

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we have

$$\vdash_{\Sigma_1} \mathscr{L}_1(P) : s,$$

so $\mathscr{L}_1(P) \in \mathcal{A}(\mathscr{G})$.

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so $\mathscr{L}_1(P) \in \mathcal{A}(\mathscr{G})$.

Since $\mathscr{L}(\mathscr{L}_1(P)) = \mathscr{L}_{\cap R}(P)$, $/a_1 \ldots a_n/ \in \mathcal{O}(\mathscr{G})$.

From (1), we also get

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Hence $|\mathscr{L}_M(P)|_\beta$ must be of the form $\lambda z.a_1^{r_1 \to q_1}(\ldots(a_n^{r_n \to q_n} z)\ldots)$. From (2), by the Subject Reduction Theorem, we obtain

$$\vdash_{\Sigma_M} \lambda z.a_1^{r_1 \to q_1}(\ldots(a_n^{r_n \to q_n} z)\ldots) : q_F \to q_I.$$

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Since $\mathscr{L}_2(\mathscr{L}_M(P)) = \mathscr{L}_{\cap R}(P) \twoheadrightarrow_\beta /a_1 \dots a_n/$, it follows that $\mathscr{L}_2(|\mathscr{L}_M(P)|_\beta) = /a_1 \dots a_n/$.

Hence $|\mathscr{L}_M(P)|_\beta$ must be of the form $\lambda z.a_1^{r_1 \to q_1}(\dots(a_n^{r_n \to q_n} z)\dots)$. From (2), by the Subject Reduction Theorem, we obtain

$$\vdash_{\Sigma_M} \lambda z.a_1^{r_1 \to q_1}(\dots(a_n^{r_n \to q_n} z)\dots) : q_F \to q_I.$$

This can only be if $q_1 = q_I$, $r_n = q_F$, and $r_i = q_{i+1}$ for $1 \leq i \leq n-1$. Since $r_i \in \delta(q_i, a_i)$, this implies that $a_1 \dots a_n \in L(M)$.

Closure under $\cap R$ (continued)

Lemma. $\mathcal{O}(\mathscr{G}) \cap L(M) \subseteq \mathcal{O}(\mathscr{G}_{\cap R})$.

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Proof.

Suppose $/a_1 \ldots a_n/ \in \mathcal{O}(\mathscr{G})$ and $a_1 \ldots a_n \in L(M)$.

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Let $P \in \mathcal{A}(\mathscr{G})$ be such that $\mathscr{L}(P) \twoheadrightarrow_\beta /a_1 \dots a_n/$.

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Let $P \in \mathcal{A}(\mathscr{G})$ be such that $\mathscr{L}(P) \twoheadrightarrow_\beta /a_1 \dots a_n/$.

Let $q_1, q_2, \dots, q_{n+1}$ be such that $q_1 = q_I$, $q_{n+1} = q_F$, and $q_{i+1} \in \delta(q_i, a_i)$ for $1 \leq i \leq n$.

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Let $P'[y_1, \ldots, y_m]$ be a constant-free linear λ-term such that $P'[c_1, \ldots, c_m] = P$, where $c_1, \ldots, c_m \in C_1$.

For $1 \leq i \leq m$, let N'_i be a constant-free linear λ-term with $\mathrm{FV}(N'_i) \subseteq \{x_1, \ldots, x_n\}$ such that

$$N'_i[a_1/x_1, \ldots, a_n/x_n] = \mathscr{L}(c_i) \quad \text{for } 1 \leq i \leq n,$$

$$P'[N'_1, \ldots, N'_m] \twoheadrightarrow_\beta \lambda z.x_1(\ldots(x_n z)\ldots).$$

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For $1 \leq i \leq n$, let $N_i = N'_i[a_1^{q_2 \rightarrow q_1}/x_1, \dots, a_n^{q_{n+1} \rightarrow q_n}/x_n]$, so that

$$\mathscr{L}_2(N_i) = \mathscr{L}(c_i). \tag{3}$$

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Then

$$P'[N_1, \ldots, N_m] \twoheadrightarrow_\beta \lambda z.a_1^{q_2 \to q_1}(\ldots(a_n^{q_{n+1} \to q_n} z)\ldots)$$

by a non-erasing non-duplicating β-reduction.

Since

$$\vdash_{\Sigma_M} \lambda z.a_1^{q_2 \to q_1}(\ldots(a_n^{q_{n+1} \to q_n} z)\ldots) : q_F \to q_I,$$

we get

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by the Subject Expansion Theorem.

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by the Subject Expansion Theorem.

Let Δ be the unique $\lambda{\to}_{\Sigma_M}$-deduction of this judgment. Δ contains a subdeduction Δ_i of

$$\vdash_{\Sigma_M} N_i : \beta_i \tag{4}$$

for some $\beta_i \in \mathscr{T}(A_M)$, for $1 \leq i \leq m$.

It is easy to see that applying the lexicon $\mathscr{L}_2$ to each step of Δ gives a $\lambda{\rightarrow}_{\Sigma_2}$-deduction Δ' of

$$\vdash_{\Sigma_2} P'[\mathscr{L}(c_1), \ldots, \mathscr{L}(c_m)] : o \rightarrow o.$$

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Since $P'[\mathscr{L}(c_1), \ldots, \mathscr{L}(c_m)] = \mathscr{L}(P)$, we see that $\mathscr{L}_2$ maps Δ_i to the unique $\lambda{\rightarrow}_{\Sigma_2}$-deduction of

$$\vdash_{\Sigma_2} \mathscr{L}(c_i) : \mathscr{L}(\tau_1(c_i)).$$

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It follows that

$$\mathscr{L}_2(\beta_i) = \mathscr{L}(\tau_1(c_i)). \qquad (5)$$

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By (3), (4), and (5),

$$d_{\langle c_i, N_i, \beta_i \rangle} \in C_{\cap R}.$$

We have

$$\{y_1 : \beta_1, \ldots, y_m : \beta_m\} \vdash P' : q_F \to q_I,$$

$$\{y_1 : \tau_1(c_1), \ldots, y_m : \tau_1(c_m)\} \vdash P' : s.$$

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Let $\tau_{\cap R}(d_{\langle c_i, N_i, \beta_i \rangle}) = \gamma_i$ for $i = 1, \dots, m$. By the definition of $\tau_{\cap R}$,

$$\langle \gamma_1, \dots, \gamma_m, s^{q_F \to q_I} \rangle$$

is a most specific common anti-instance of

$$\langle \beta_1, \dots, \beta_m, q_F \to q_I \rangle \quad \text{and} \quad \langle \tau_1(c_1), \dots, \tau_1(c_m), s \rangle.$$

By the Principal Pair Theorem, it follows that

$$\{y_1 : \gamma_1, \ldots, y_m : \gamma_m\} \vdash P' : s^{q_F \to q_I}$$

and hence

$$\vdash_{\Sigma \cap R} P'[d_{\langle c_1, N_1, \beta_1 \rangle}, \ldots, d_{\langle c_m, N_m, \beta_m \rangle}] : s^{q_F \to q_I}.$$

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Therefore, $P'[d_{\langle c_1, N_1, \beta_1 \rangle}, \ldots, d_{\langle c_m, N_m, \beta_m \rangle}] \in \mathcal{A}(\mathscr{G}_{\cap R})$.

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Therefore, $P'[d_{\langle c_1, N_1, \beta_1 \rangle}, \ldots, d_{\langle c_m, N_m, \beta_m \rangle}] \in \mathcal{A}(\mathscr{G}_{\cap R})$.

$$\begin{aligned} & \mathscr{L}_{\cap R}(P'[d_{\langle c_1, N_1, \beta_1 \rangle}, \ldots, d_{\langle c_m, N_m, \beta_m \rangle}]) \\ & \quad = P'[\mathscr{L}_{\cap R}(d_{\langle c_1, N_1, \beta_1 \rangle}), \ldots, \mathscr{L}_{\cap R}(d_{\langle c_m, N_m, \beta_m \rangle})] \\ & \quad = P'[\mathscr{L}(c_1), \ldots, \mathscr{L}(c_m)] \\ & \quad = \mathscr{L}(P) \\ & \quad \twoheadrightarrow_\beta /a_1 \ldots a_n/. \end{aligned}$$

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Theorem. $\mathcal{O}(\mathscr{G}_{\cap R}) = \mathcal{O}(\mathscr{G}) \cap \{\, /w/ \mid w \in L(M) \,\}$.

Closure under h^{-1}

Lemma.

The string languages of ACGs are closed under substitution.

$$a \mapsto \mathcal{O}(\mathscr{G})$$

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Fact.

If a family of languages includes the regular sets and is closed under substitution and $\cap R$, then it is closed under h^{-1}.

ACGs give rise to full AFLs

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The string languages of ACGs in $\mathbf{G}(m, n)$ ($m \geq 2$) form a full AFL.

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If a family of ϵ-free languages includes the ϵ-free regular sets and is closed under substitution, $\cap R$, and k-limited erasing, then it is closed under h^{-1}.

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Lemma.

The string languages of lexicalized ACGs are closed under k-limited erasing.

Theorem.

The string languages of lexicalized ACGs form an AFL.