Toward a Logic of Cumulative Quantification

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Scha (1984), "Distributive, Collective and Cumulative Quantification"

sentences with indefinite noun phrases may have readings which cannot be represented by means of a formula which has one quantifier for every noun phrase – for instance, when (4a) is read as (4b).

(4a) 600 Dutch firms use 5000 American computers.
(4b) The total number of Dutch firms that use an American computer is 600 and the total number of American computers used by a Dutch firm is 5000.

This phenomenon has been called *cumulative quantification* (cf. Scha, 1978). In order to generate cumulative readings, our grammar can translate a sequence of noun phrases into one single quantifier, ranging over the cartesian product of the extensions of the nouns.

The notion of cumulative quantification was first introduced by Remko Scha.
Distinct from subject-wide-scope and object-wide-scope readings.

Cumulation in Generalized Quantifier Theory

600 Dutch firms use 5000 American computers.
Three boys kissed five girls.

$$\text{Cum}(Q_1, Q_2)\ R \Leftrightarrow Q_1 x\ \exists y\ R(x,y) \wedge Q_2 y\ \exists x\ R(x,y)$$

$$[\![n\ \text{N}]\!] = \{\ Y \mid |[\![\text{N}]\!] \cap Y| = n\ \}$$

Cum takes two type <1> quantifiers and returns a type <2> quantifier.

Meanings of Numerals

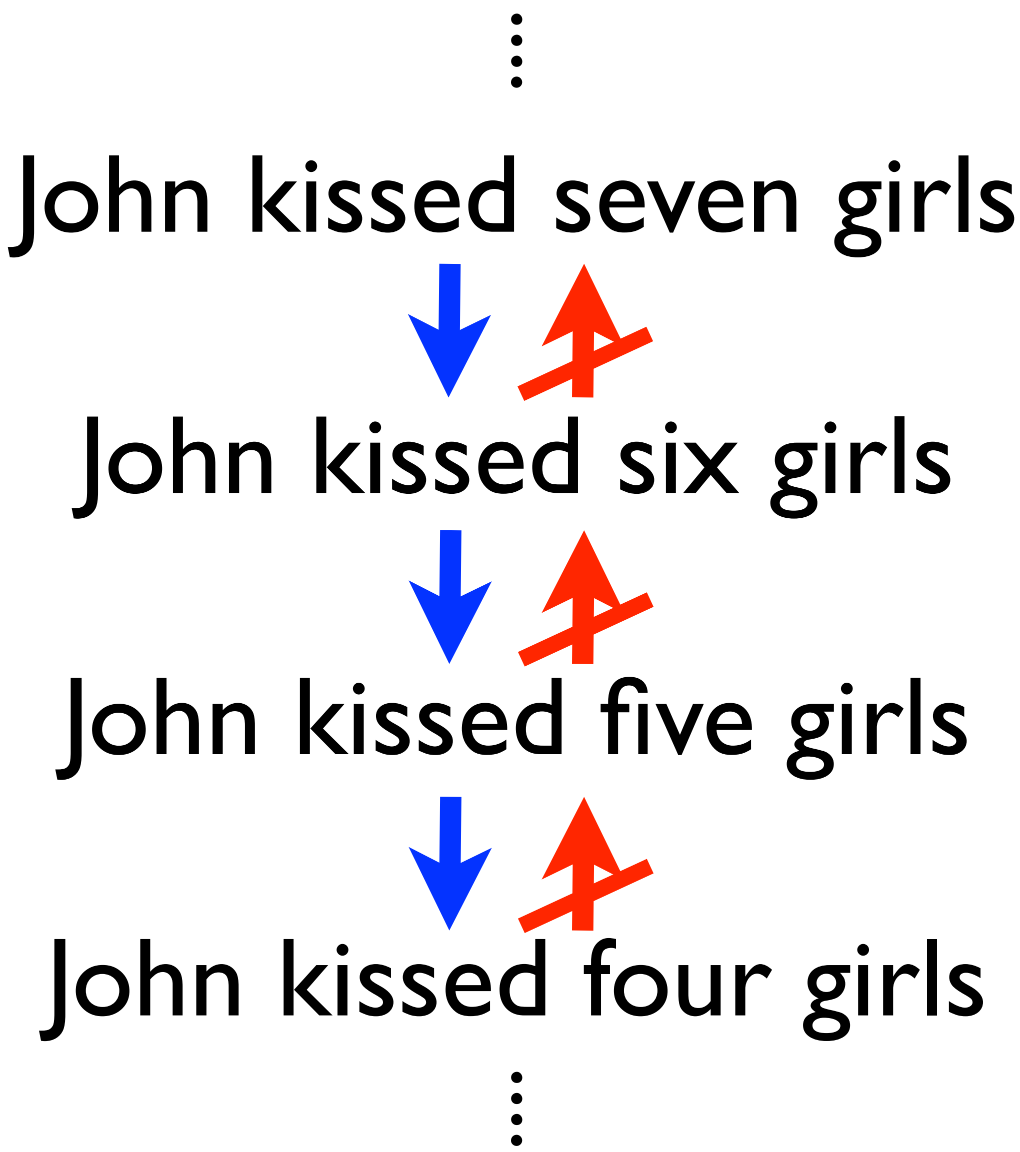
John kissed five girls

$|[\![\text{girl}]\!] \cap \{ x \mid [\![\text{kissed}]\!](\text{John}, x) \}| = 5$

$|[\![\text{girl}]\!] \cap \{ x \mid [\![\text{kissed}]\!](\text{John}, x) \}| \geq 5$

"John kissed five girls" is consistent with "John kissed more than five girls".

Scalar Implicatures



"John kissed five girls" implicates "¬(John kissed six girls)"

Stronger alternatives to the utterance are negated.

Meanings of Numerals

John kissed five girls

$$|[\![\text{girl}]\!] \cap \{ x \mid [\![\text{kissed}]\!](\text{John}, x) \}| = 5$$

$$|[\![\text{girl}]\!] \cap \{ x \mid [\![\text{kissed}]\!](\text{John}, x) \}| \geq 5$$

$$[\![n \text{ N}]\!] = \{ Y \mid |[\![\text{N}]\!] \cap Y| \geq n \}$$

$$\exists X \subseteq [\![\text{girl}]\!](|X| = 5 \wedge {}^{**}[\![\text{kissed}]\!](\text{John}, X))$$

$$[\![n \text{ N}]\!] = \{ \mathbf{Y} \mid \exists \text{X} \subseteq [\![\text{N}]\!](|X| = n \wedge \text{X} \in \mathbf{Y})\}$$

Meanings of Cumulative Sentences

$[\![m \text{ N}_1 \text{ V } n \text{ N}_2]\!] \Leftrightarrow$

$\exists X \subseteq [\![\text{N}_1]\!](|X| = m \wedge \exists Y \subseteq [\![\text{N}_2]\!](|Y| = n \wedge {}^{**}[\![\text{V}]\!](X, Y)))$

$^{**}R(X, Y) \Leftrightarrow \forall x \in X\ \exists y \in Y\ R(x, y) \wedge \forall y \in Y\ \exists x \in X\ R(x, y)$

$\Leftrightarrow \exists R' \subseteq R(X = \text{dom}(R') \wedge Y = \text{ran}(R'))$

Krifka 1999, Landman 2000

**R is the “cumulative closure” of R.

Meanings and Implicatures(?)

Krifka-Landman semantics amounts to:

$$[\![m\ \mathrm{N}_1\ \ \mathrm{V}\ \ n\ \mathrm{N}_2]\!]$$

$$\Leftrightarrow \exists R' \subseteq R(|\mathrm{dom}(R')| = m \wedge |\mathrm{ran}(R')| = n)$$

$$\text{where } R = [\![\mathrm{V}]\!] \cap ([\![\mathrm{N}_1]\!] \times [\![\mathrm{N}_2]\!])$$

which is weaker than Scha's truth conditions:

$$|\mathrm{dom}(R)| = m \wedge |\mathrm{ran}(R)| = n$$

Pragmatic Scale for Cumulative Sentences

⋮			⋮	
six			eight	
five			seven	
four			six	
three	boys	kissed	five	girls
two			four	
one			three	
			⋮	

Abbreviation: Write $R(m, n)$ for

$$\exists R' \subseteq R(|\mathrm{dom}(R')| = m \wedge |\mathrm{ran}(R')| = n)$$

What is said:

$$[\![m\ \mathrm{N}_1\ \ \mathrm{V}\ \ n\ \mathrm{N}_2]\!] \Leftrightarrow R(m, n)$$

$$\text{where } R = [\![\mathrm{V}]\!] \cap ([\![\mathrm{N}_1]\!] \times [\![\mathrm{N}_2]\!])$$

Krifka-Landman implicature:

$$\forall m' \forall n' (R(m', n') \rightarrow (m' < m \vee n' < n \vee (m' = m \wedge n' = n)))$$

or, equivalently,

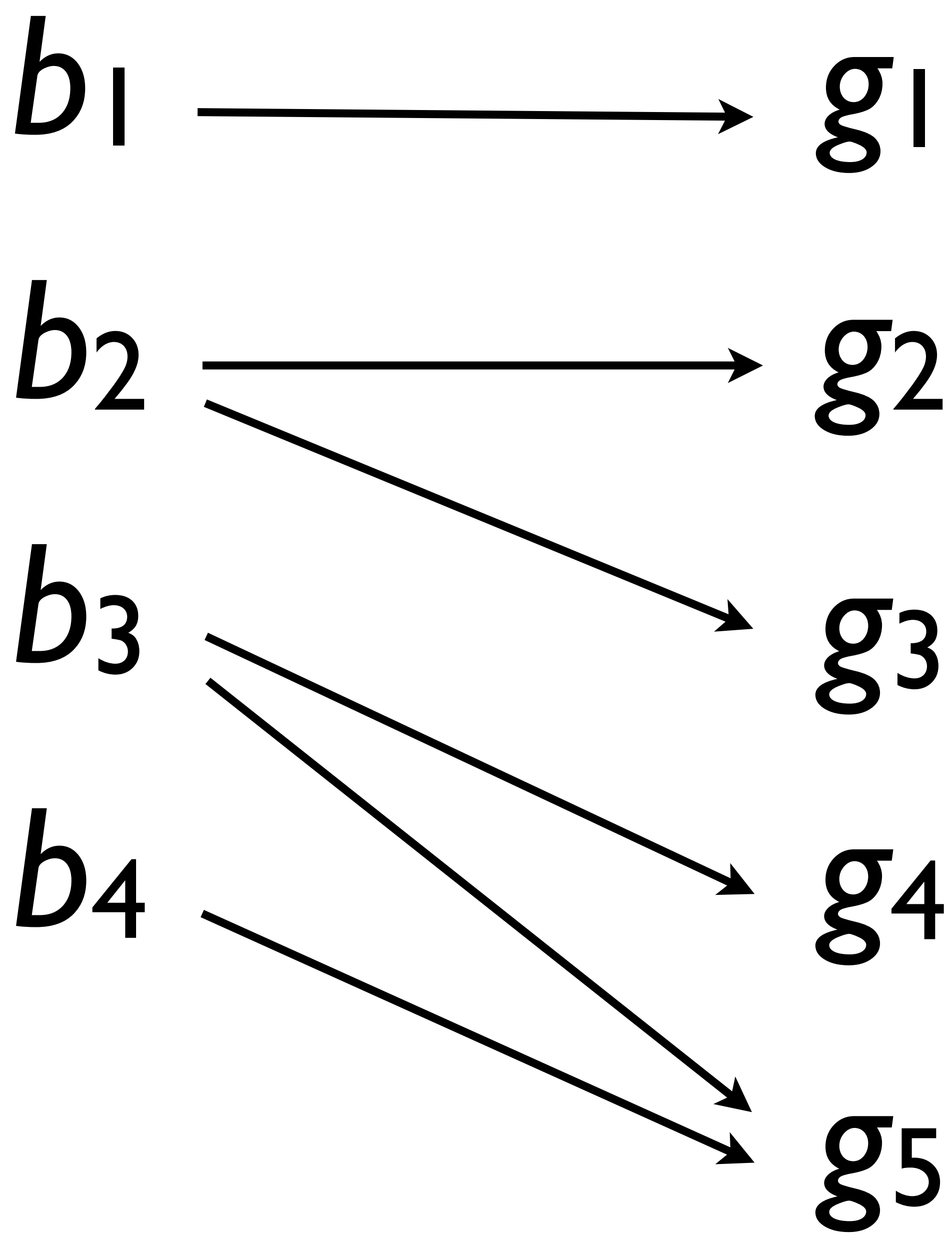
$$\forall m' \forall n' (R(m', n') \rightarrow (m' \leq m \wedge n' \leq n))$$

The idea is that if $m' \geq m \wedge n' \geq n \wedge \neg(m' = m \wedge n' = n)$, then $R(m',n')$ is “higher up” in the scale than $R(m,n)$, so the assertion of $R(m,n)$ implicates $\neg R(m',n')$. By taking contrapositive, we get $R(m',n') \rightarrow m' < m \vee n' < n \vee (m' = m \wedge n' = n)$, which is the first formulation of the Krifka-Landman implicature.

To see that the first formulation implies the second, suppose $R(m,n) \wedge R(m',n')$. Then **R(X,Y) and **R(X′,Y′) with $|X| = m$, $|Y| = n$, $|X'| = m'$, $|Y'| = n'$. From this we get **R($X \cup X'$,$Y \cup Y'$), so $R(m'',n'')$ with $m'' \geq \max(m,m')$ and $n'' \geq \max(n,n')$. If $m' > m$, then $m'' > m$ and $n'' \geq n$, so $\neg R(m'',n'')$ should be an implicature, contradicting $R(m'',n'')$. So we must have $m' \leq m$. Similarly, we can derive $n' \leq n$.

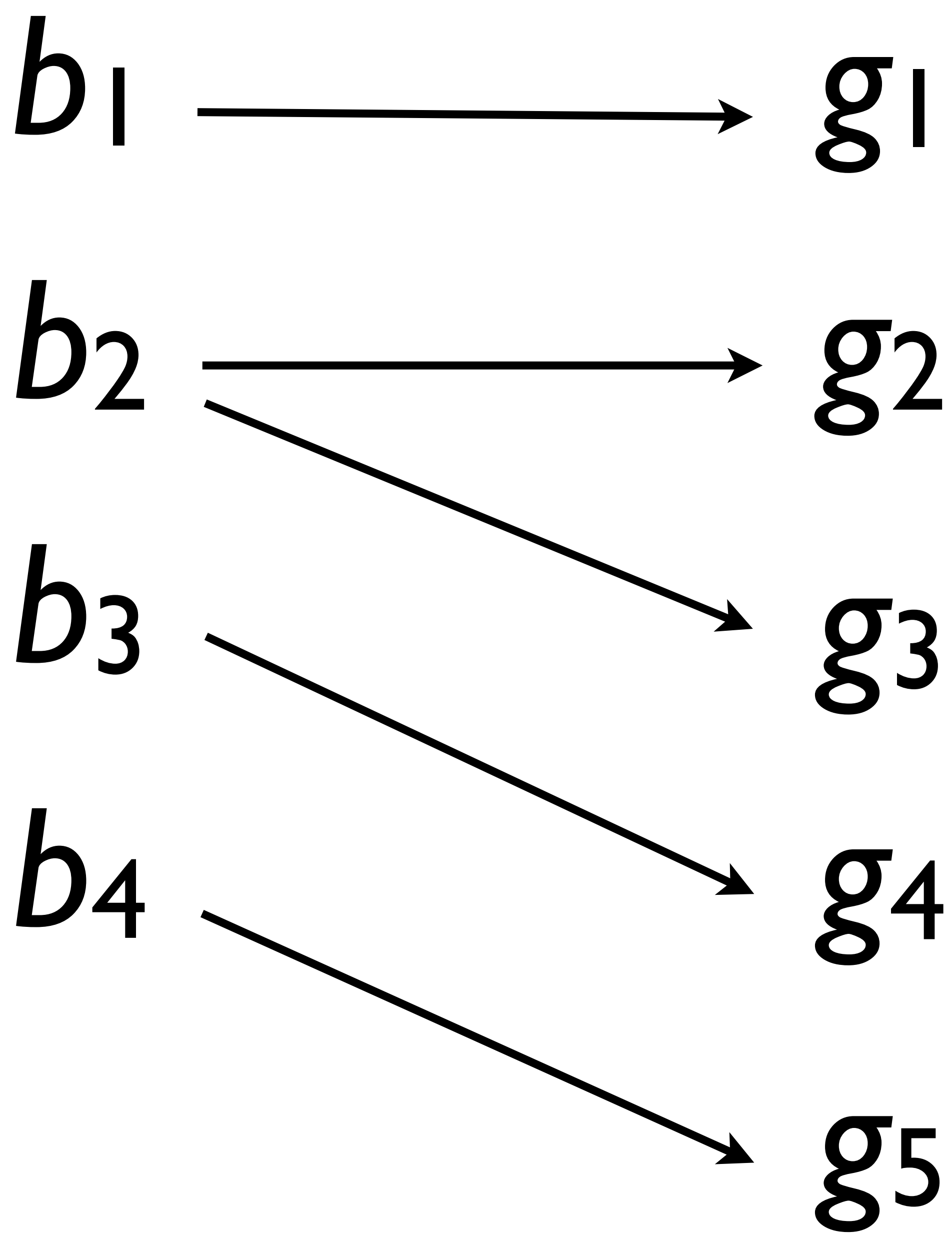
This shows that in the presence of $R(m,n)$, the two formulations of the implicature are equivalent.

Three boys kissed five girls



To be maximally informative, should say "Four boys kissed five girls".

Four boys kissed five girls



four boys kissed five girls $\not\models$ three boys kissed five girls

$$R(m, n) \Leftrightarrow \exists R' \subseteq R(|\mathrm{dom}(R')| = m \wedge |\mathrm{ran}(R')| = n)$$

$$m\ \mathrm{N}_1\ \ \mathrm{V}\ \ n\ \mathrm{N}_2 \vDash k\ \mathrm{N}_1\ \ \mathrm{V}\ \ l\ \mathrm{N}_2$$

$$\Updownarrow$$

$$R(m, n) \vDash R(k, l)$$

When does $R(m, n)$ logically imply $R(k, l)$?

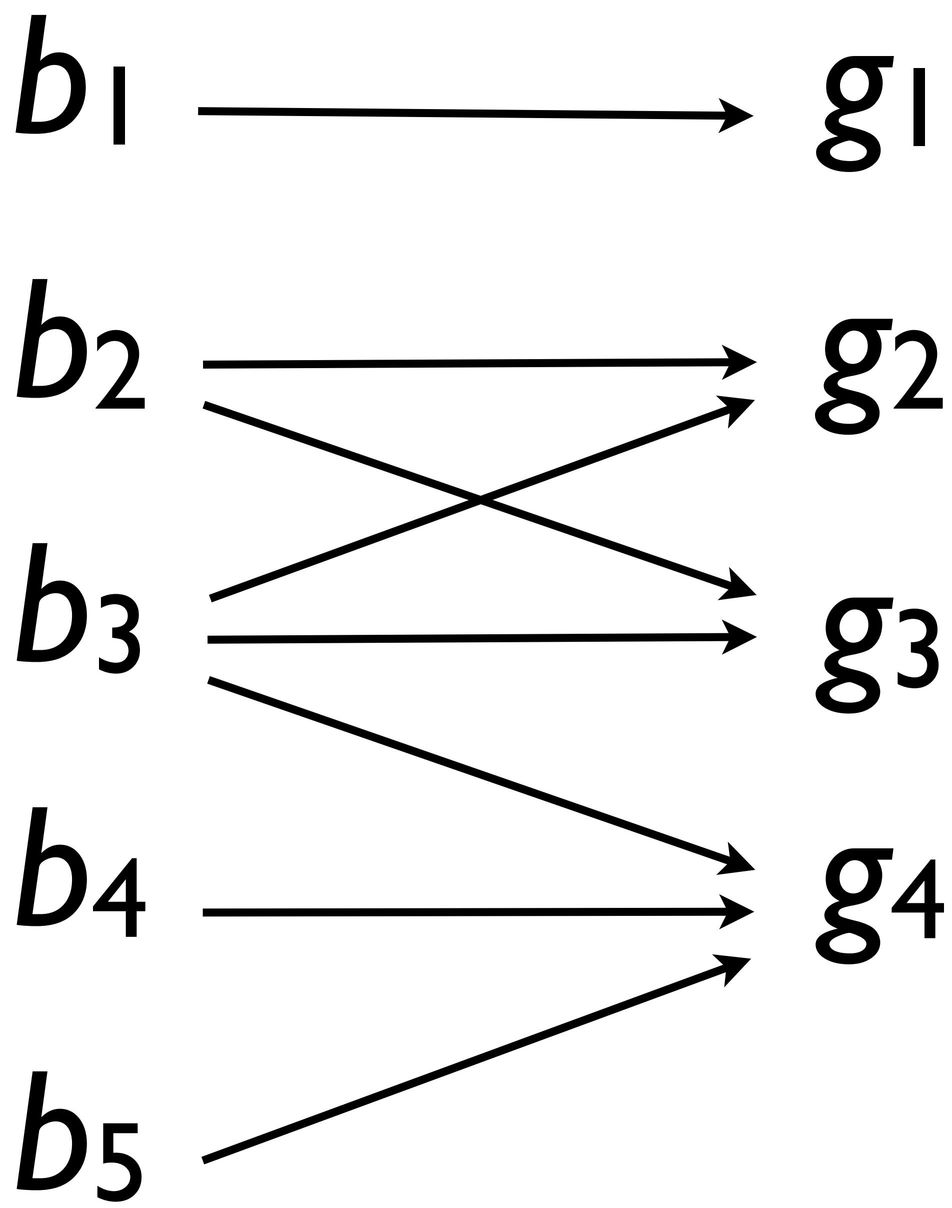
Deterministic Reducts

- If R is a binary relation, its *deterministic reduct* is:

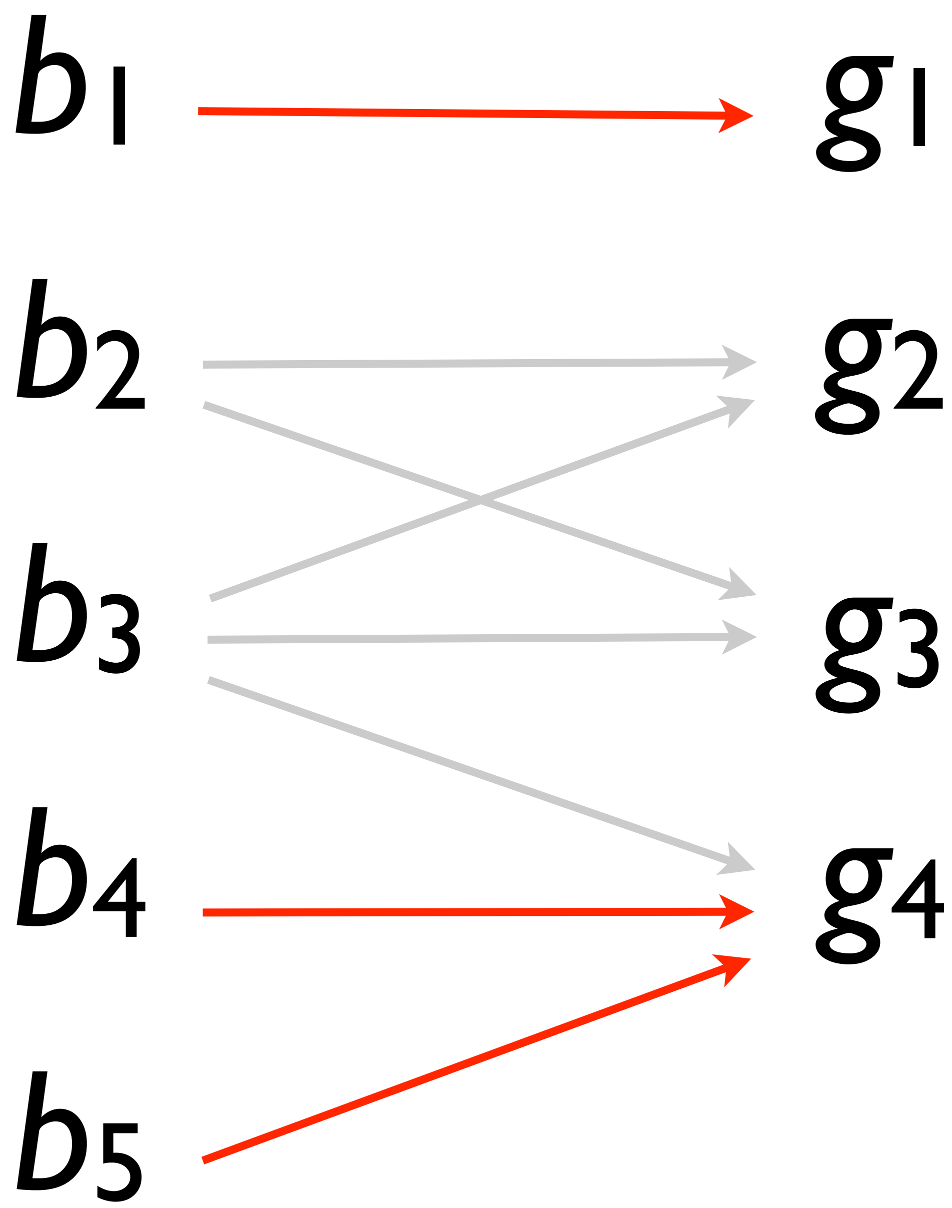
$$d(R) = \{ (x, y) \in R \mid \forall y' ((x, y') \in R \rightarrow y' = y) \}.$$

- $(x, y) \in d(R)$ means that y is the only element related to x by R.
- $d(R)$ is a partial function.

Deterministic Reducts



Deterministic Reducts

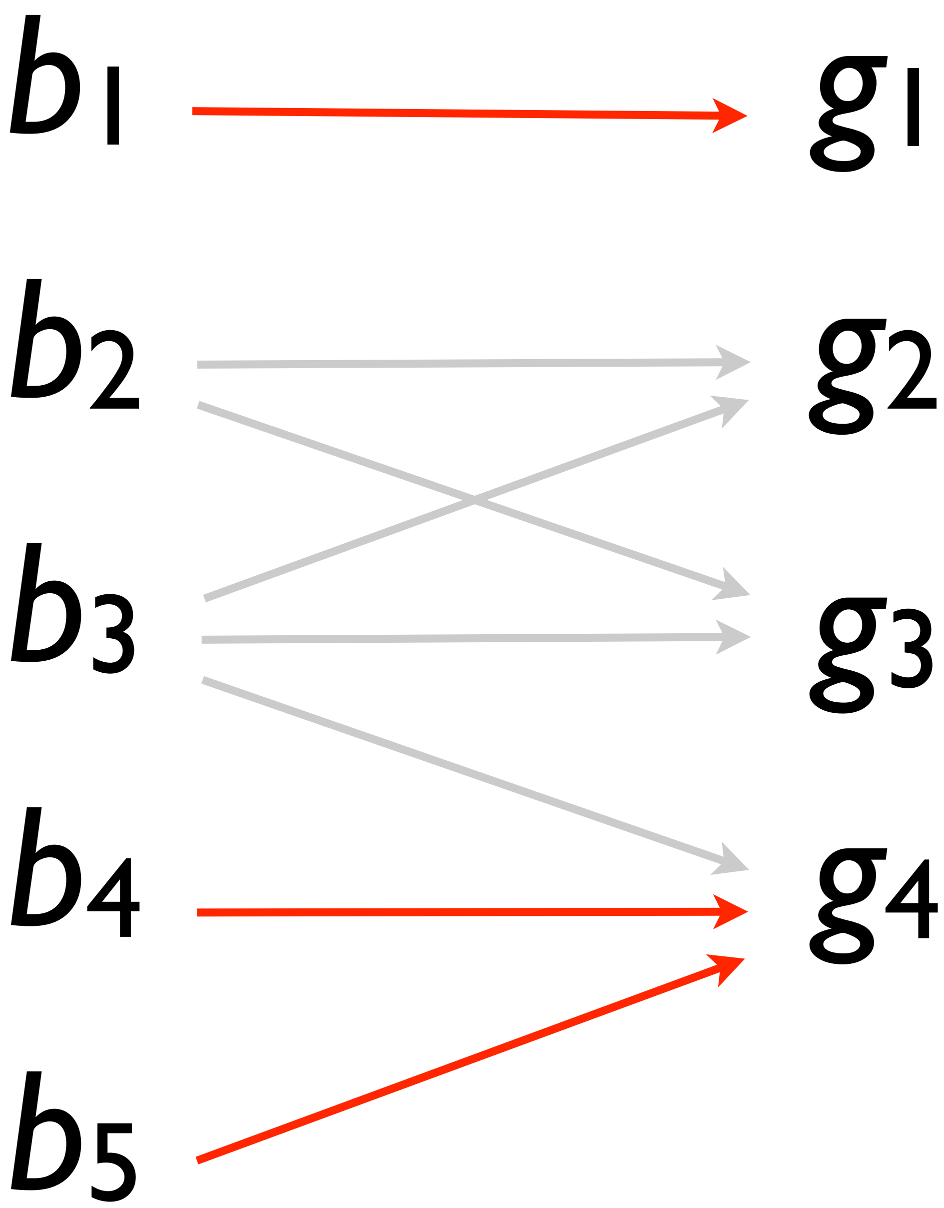


Deterministic Reducts

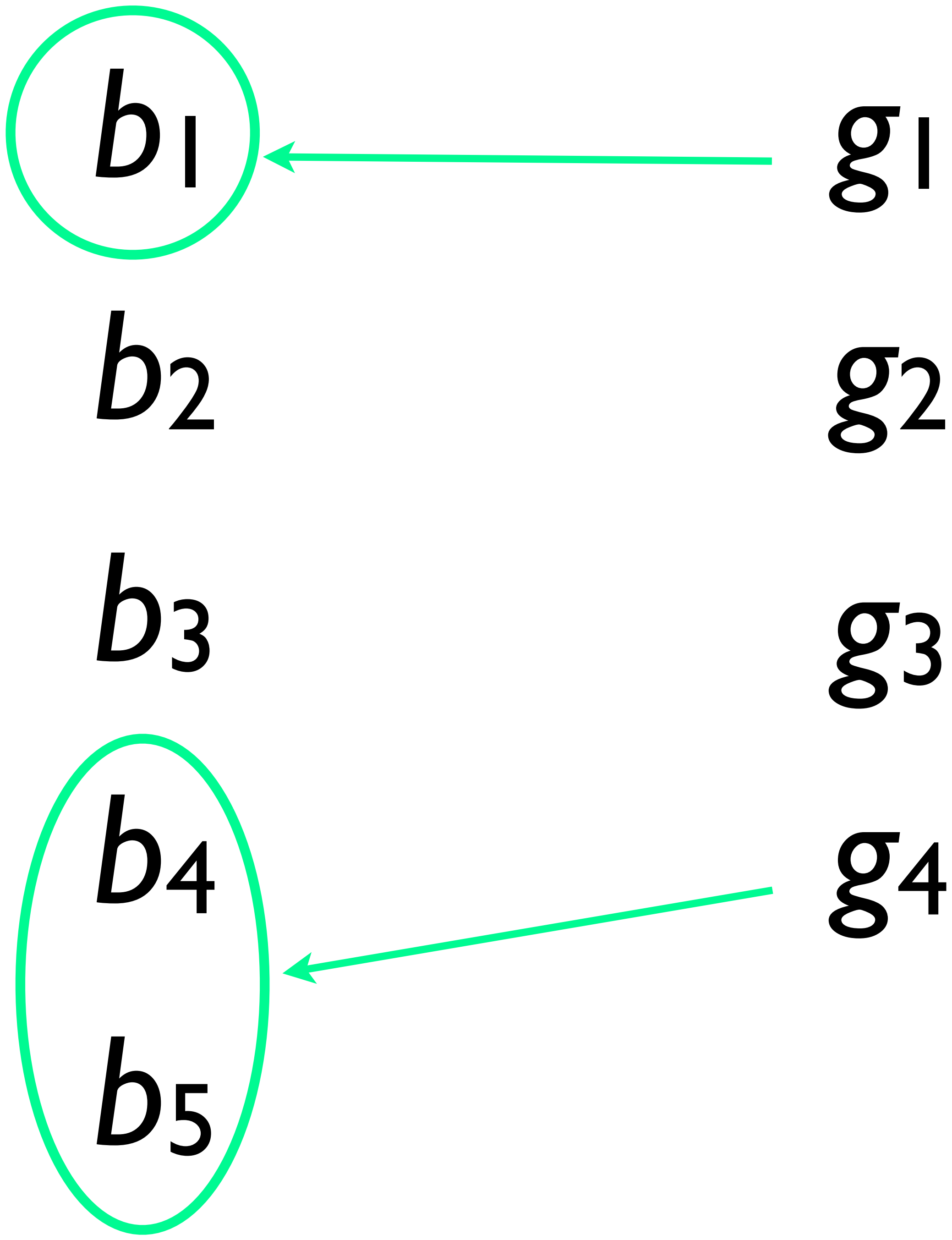
- $(d(R))^{-1}$ is the inverse image of $d(R)$:

$$(d(R))^{-1}(y) = \{ x \mid (x, y) \in d(R) \}.$$

$d(R)$



$(d(R))^{-1}$



$(d(R))^{-1}(g_1) = \{b_1\}$

$(d(R))^{-1}(g_2) = \varnothing$

$(d(R))^{-1}(g_3) = \varnothing$

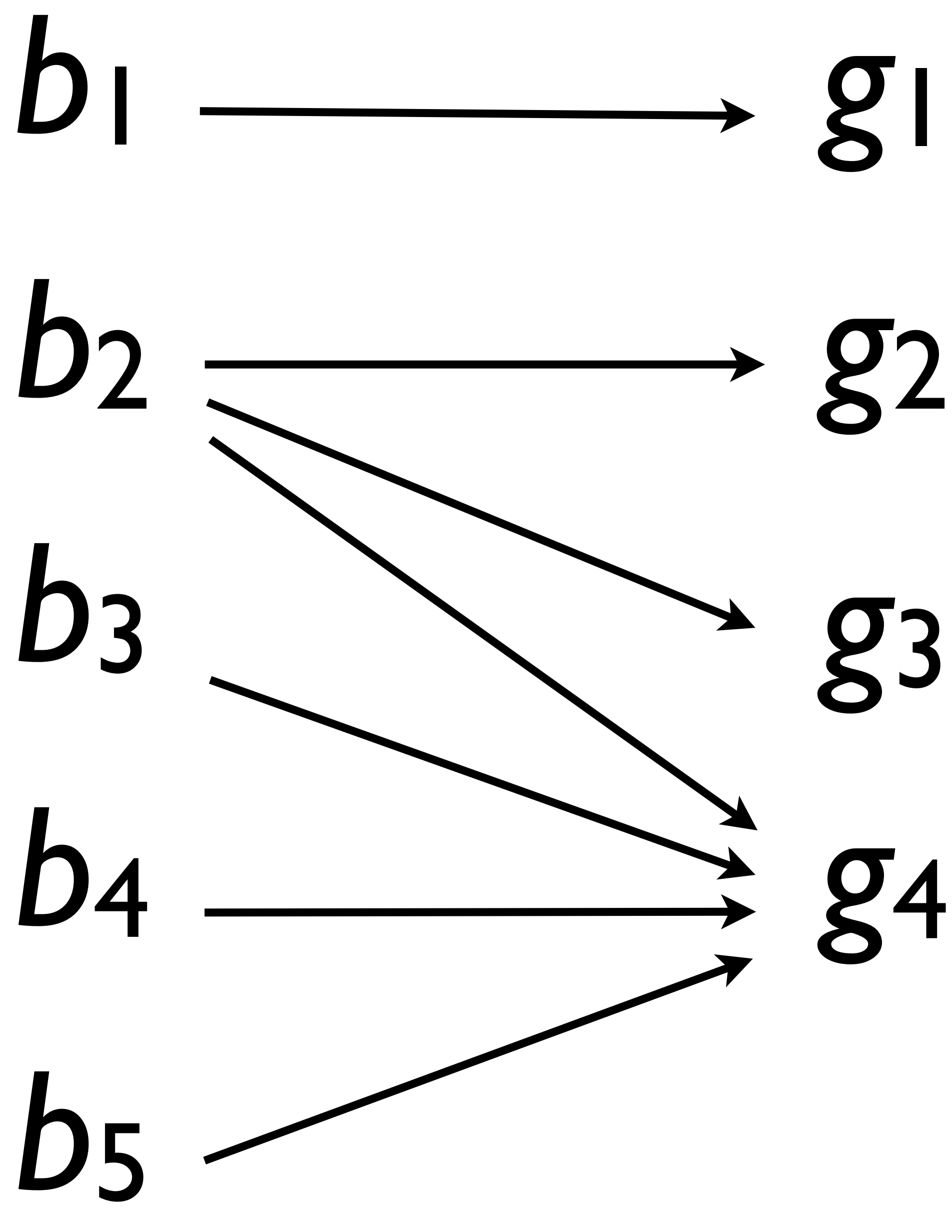
$(d(R))^{-1}(g_4) = \{b_4, b_5\}$

Entailments

Lemma 11

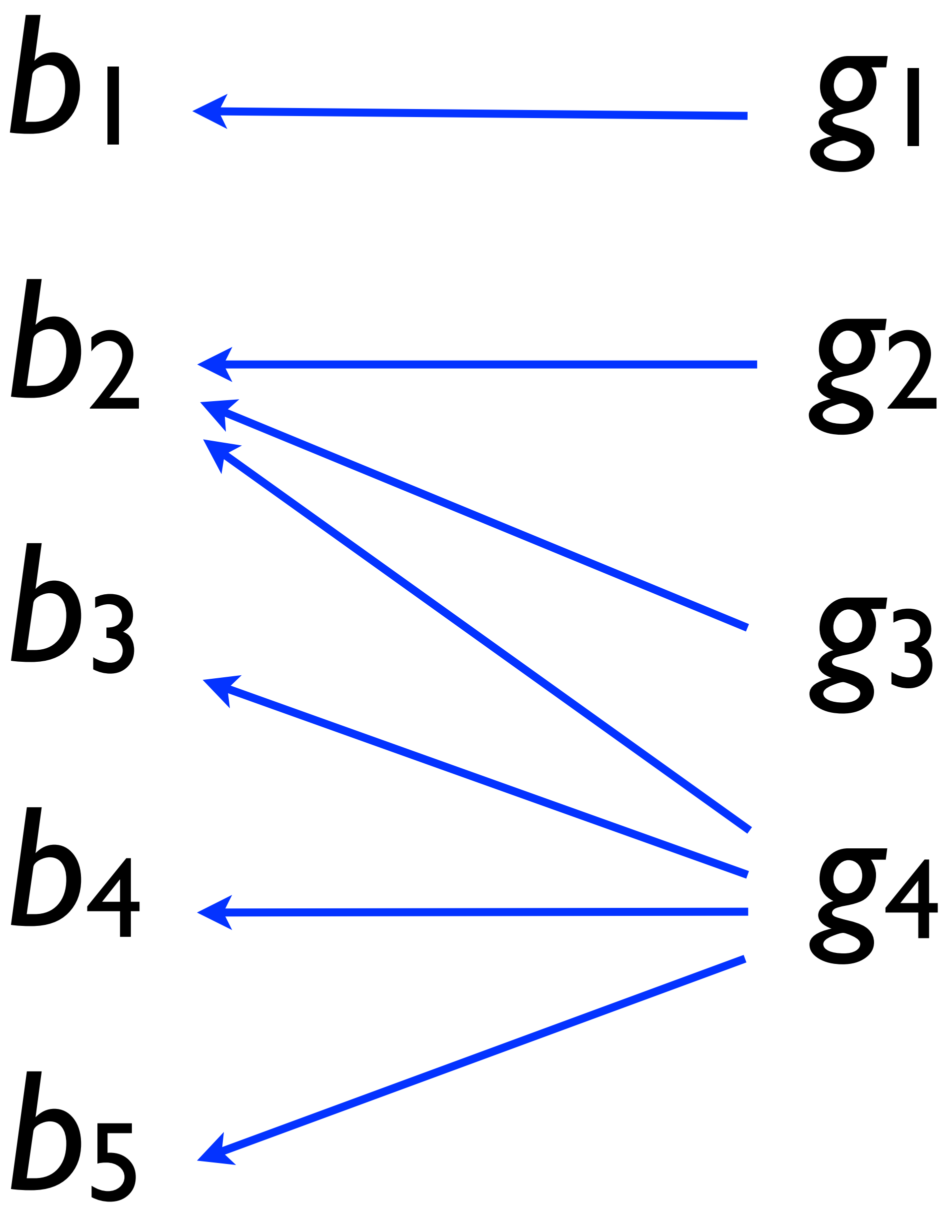
If $m > n$, then $R(m, n) \models R(m-1, n)$.

R

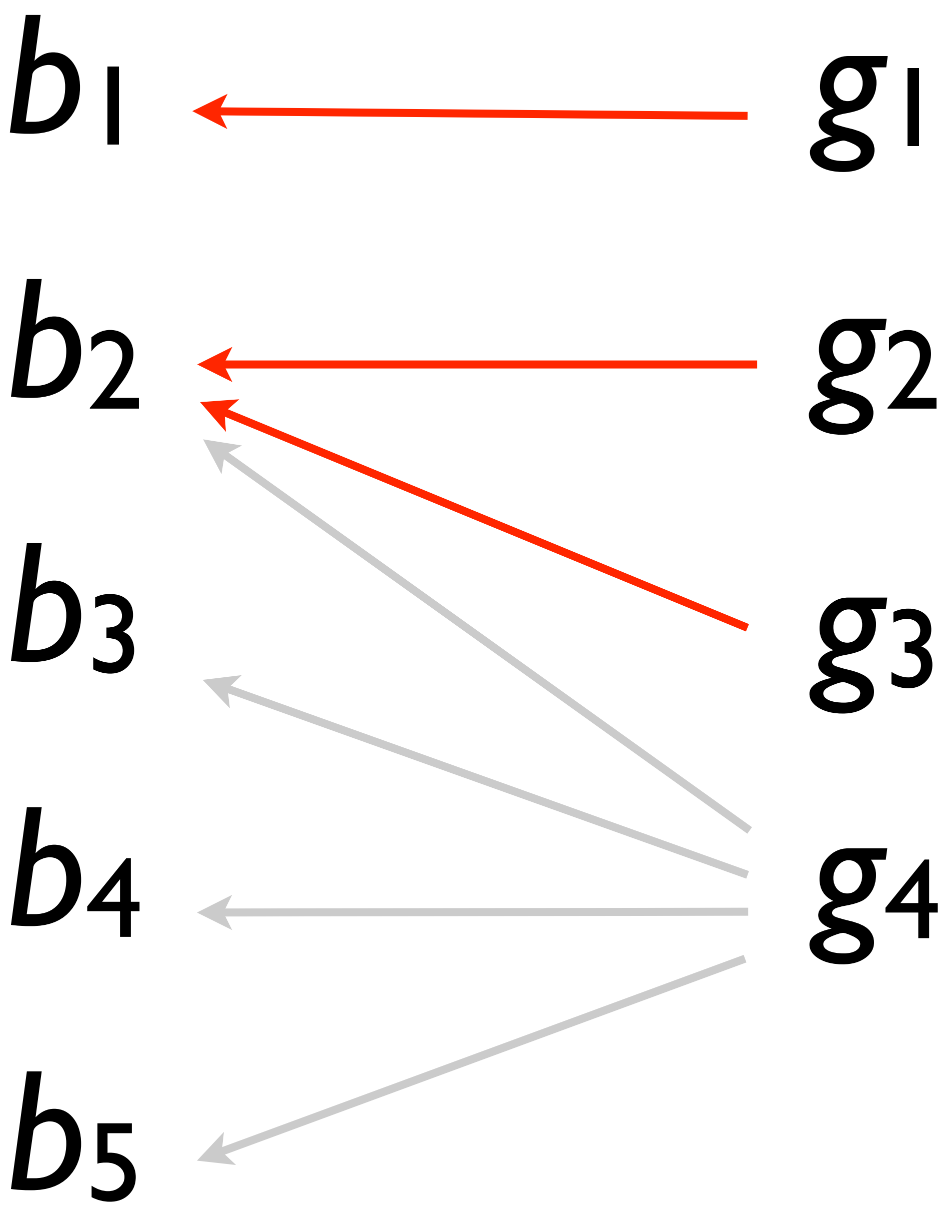


- Suppose
  B = dom(R),
  G = ran(R),
  $|B| = m$,
  $|G| = n$,
  so $R(m, n)$.

R^{-1}

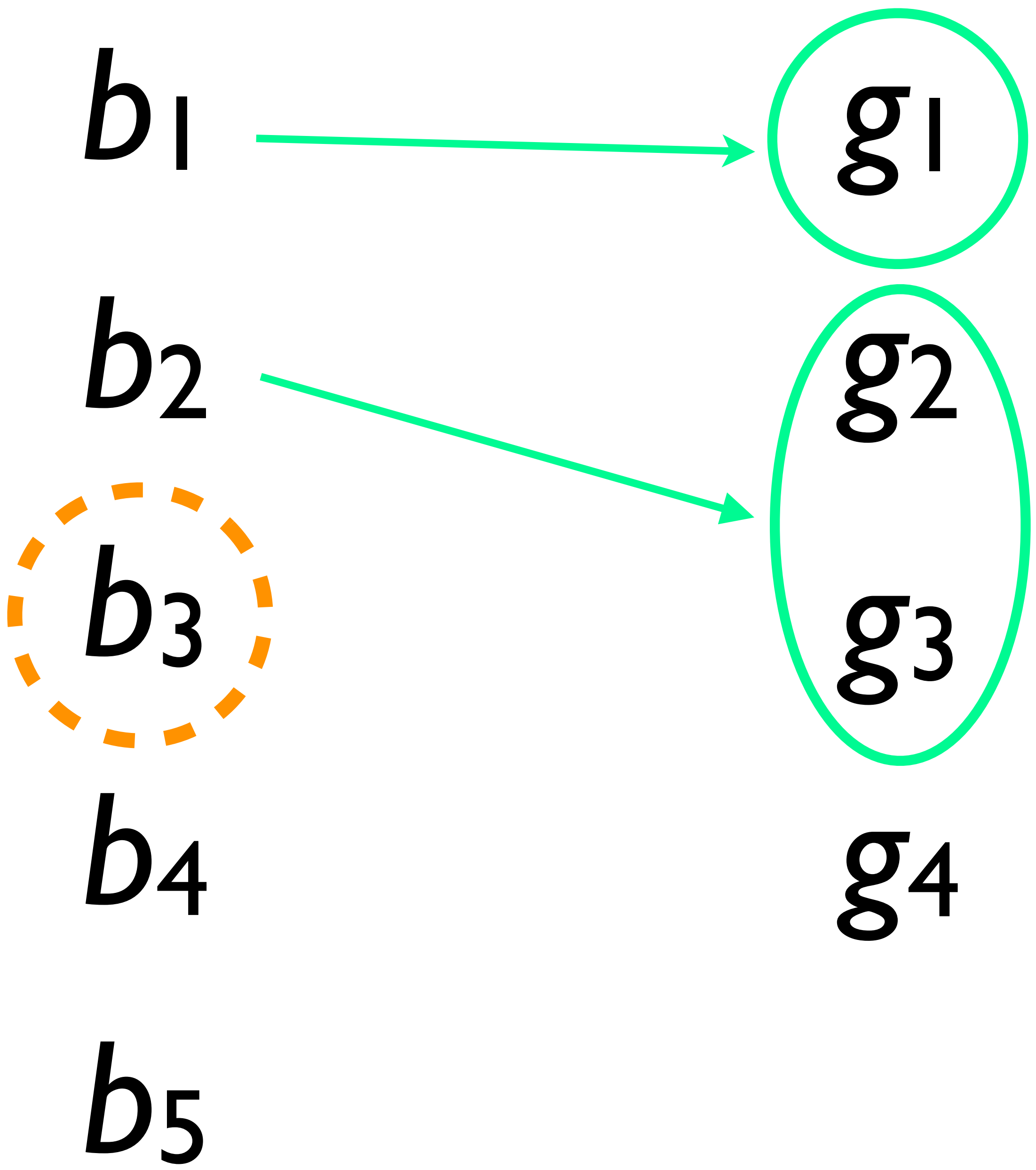


$d(R^{-1})$



- Since $d(R^{-1})$ is a partial function from G to B and since $|B| > |G|$...

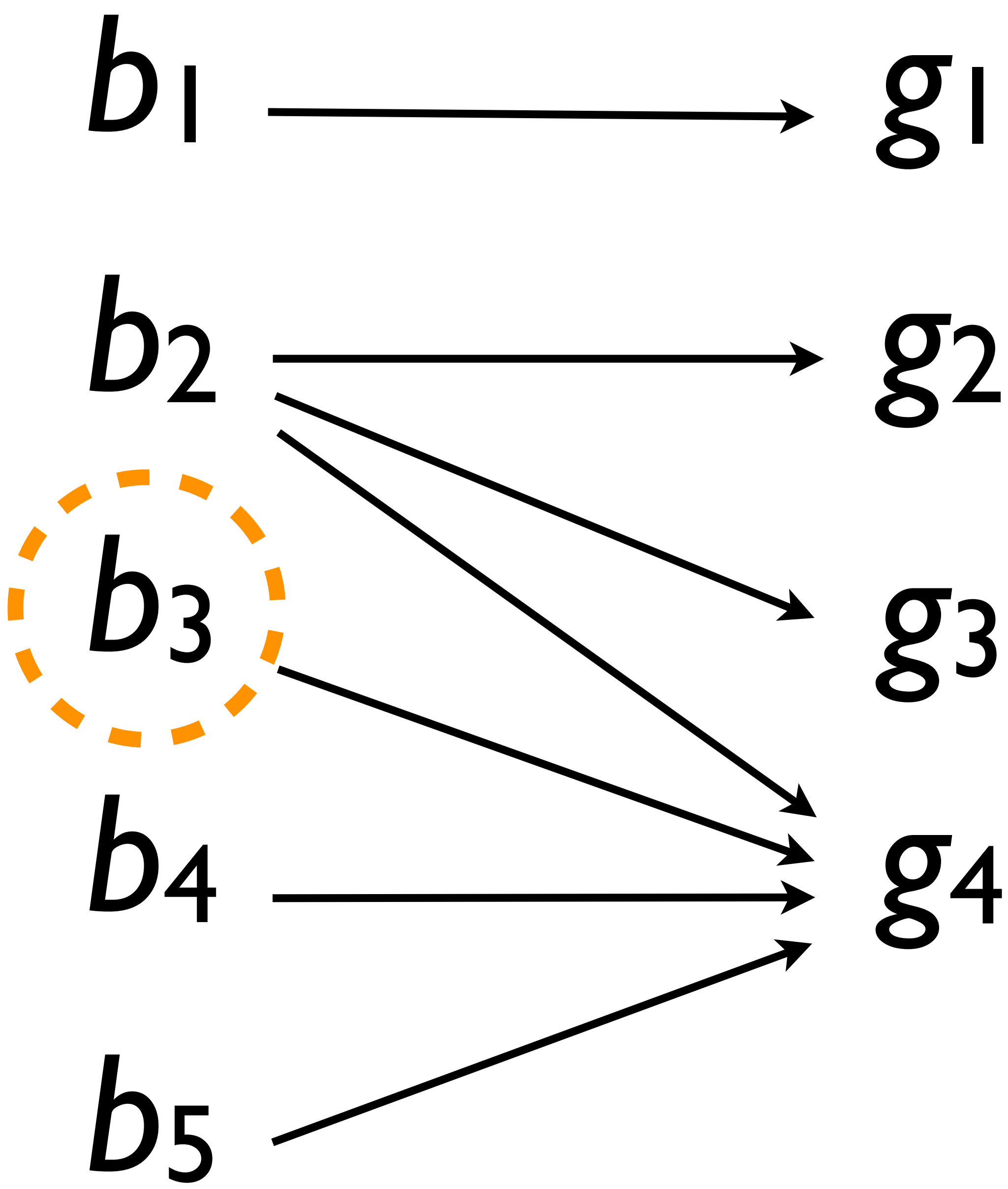
$(d(R^{-1}))^{-1}$



- ...there must be some $b \in B$ s.t. $(d(R^{-1}))^{-1}(b) = \varnothing$.

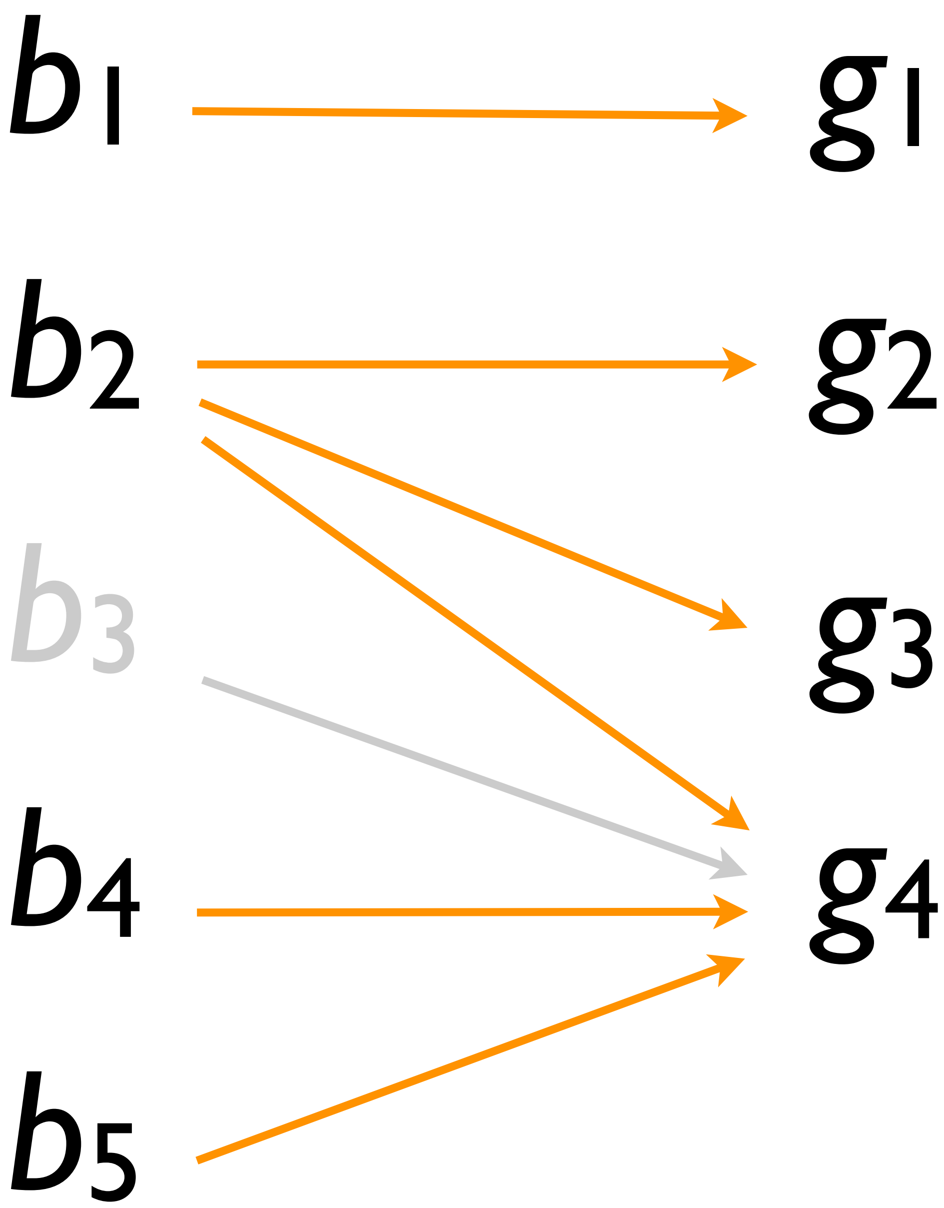
  (Here, we have $(d(R^{-1}))^{-1}(b_3) = \varnothing$ for instance.)

R



- Think of the relation R' obtained from R by removing that element b (= b_3 in this case).

R'



- Since $R'(m-1, n)$, $R(m-1, n)$. QED.

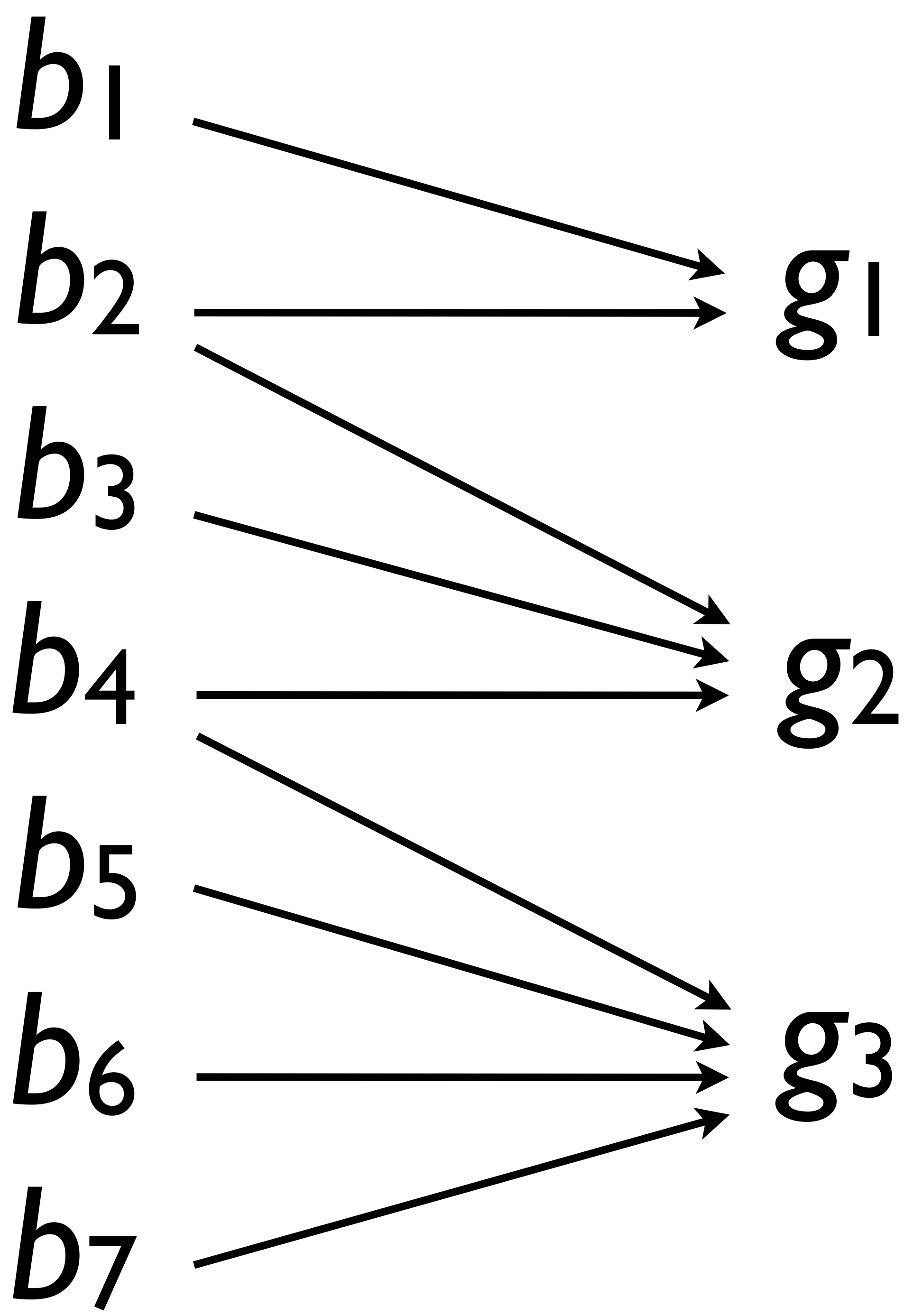
Entailments

Lemma 12

If $m \geq n > 1$, then $R(m, n) \models R(m - \lfloor m/n \rfloor, n - 1)$.

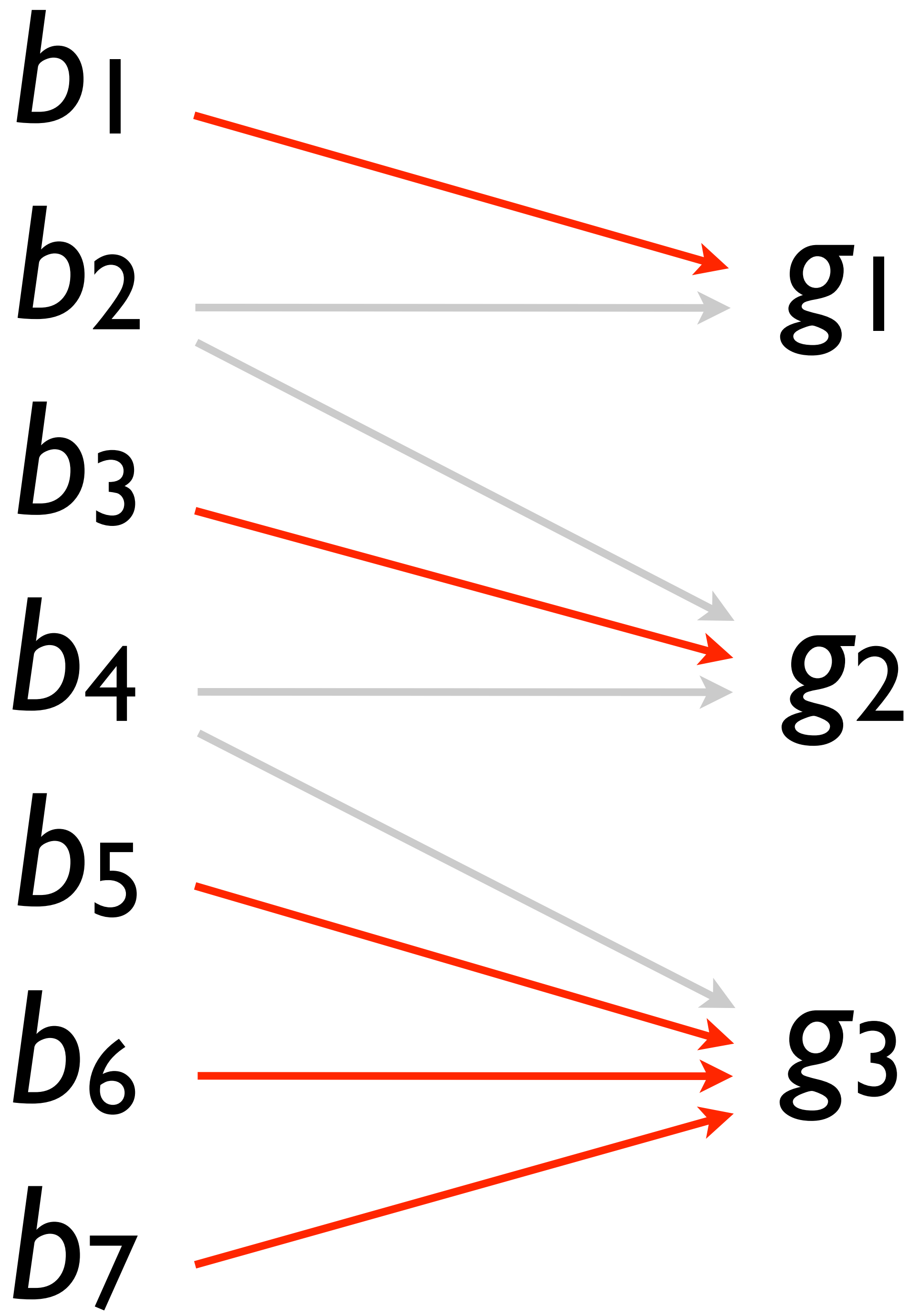
Here, $\lfloor m/n \rfloor$ denotes the quotient of m divided by n (e.g., $\lfloor 12/5 \rfloor = 2$).

R

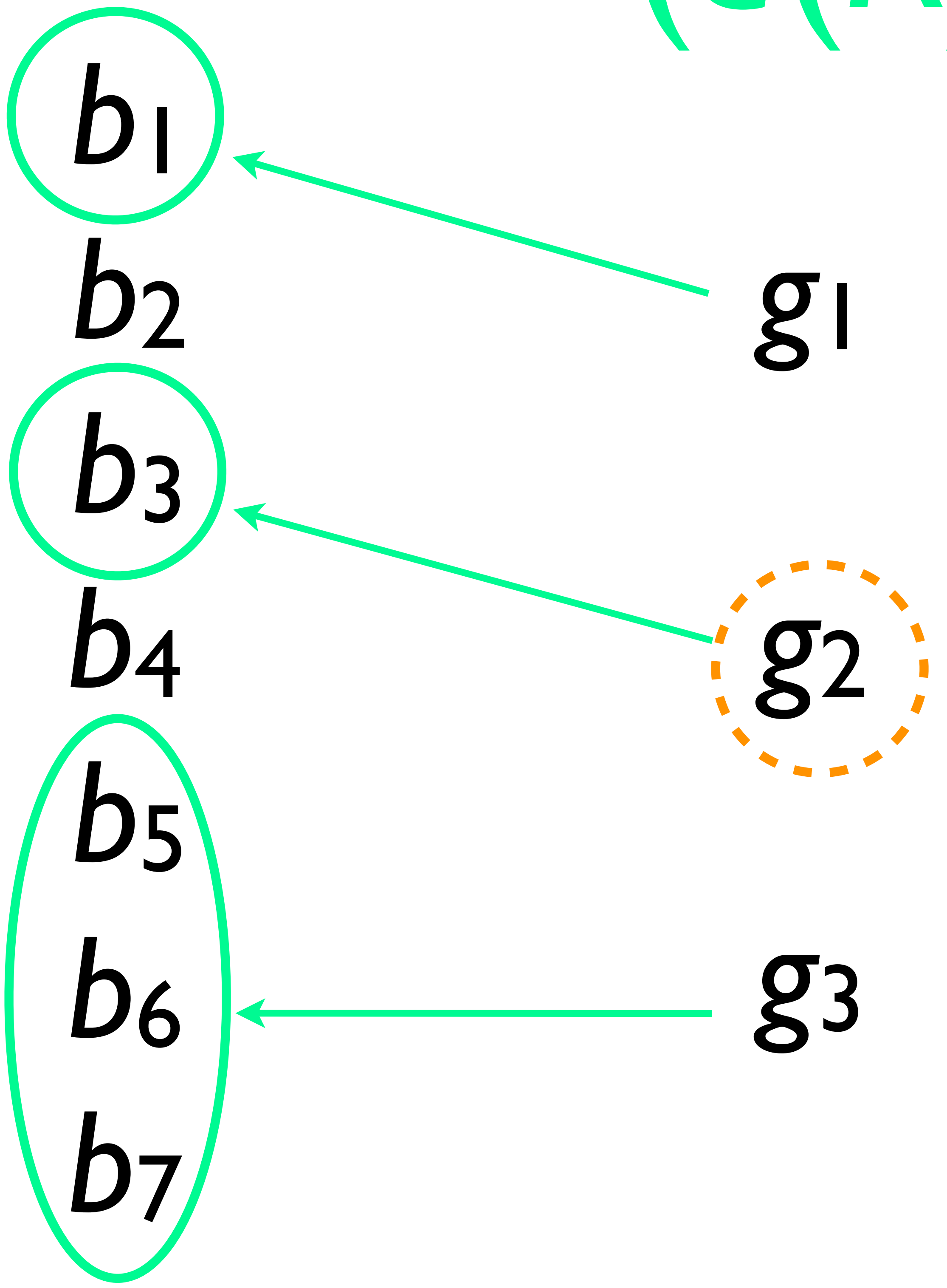


- Suppose
  $B = \mathrm{dom}(R)$,
  $G = \mathrm{ran}(R)$,
  $|B| = m$,
  $|G| = n$,
  so $R(m, n)$.

$d(R)$



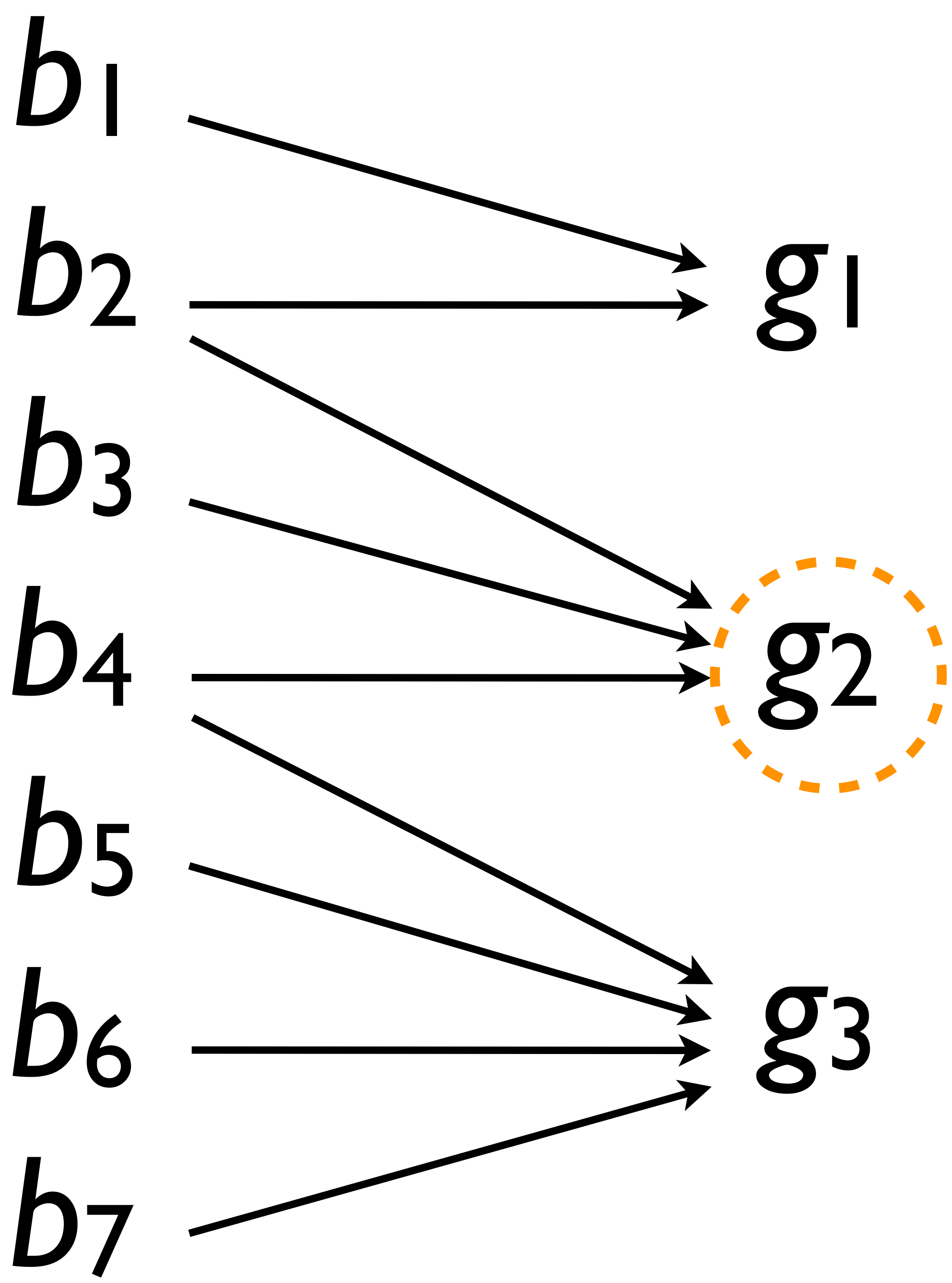
$(d(R))^{-1}$



- There must be some $g \in G$ s.t. $|(d(R))^{-1}(g)| \leq \lfloor m/n \rfloor$.

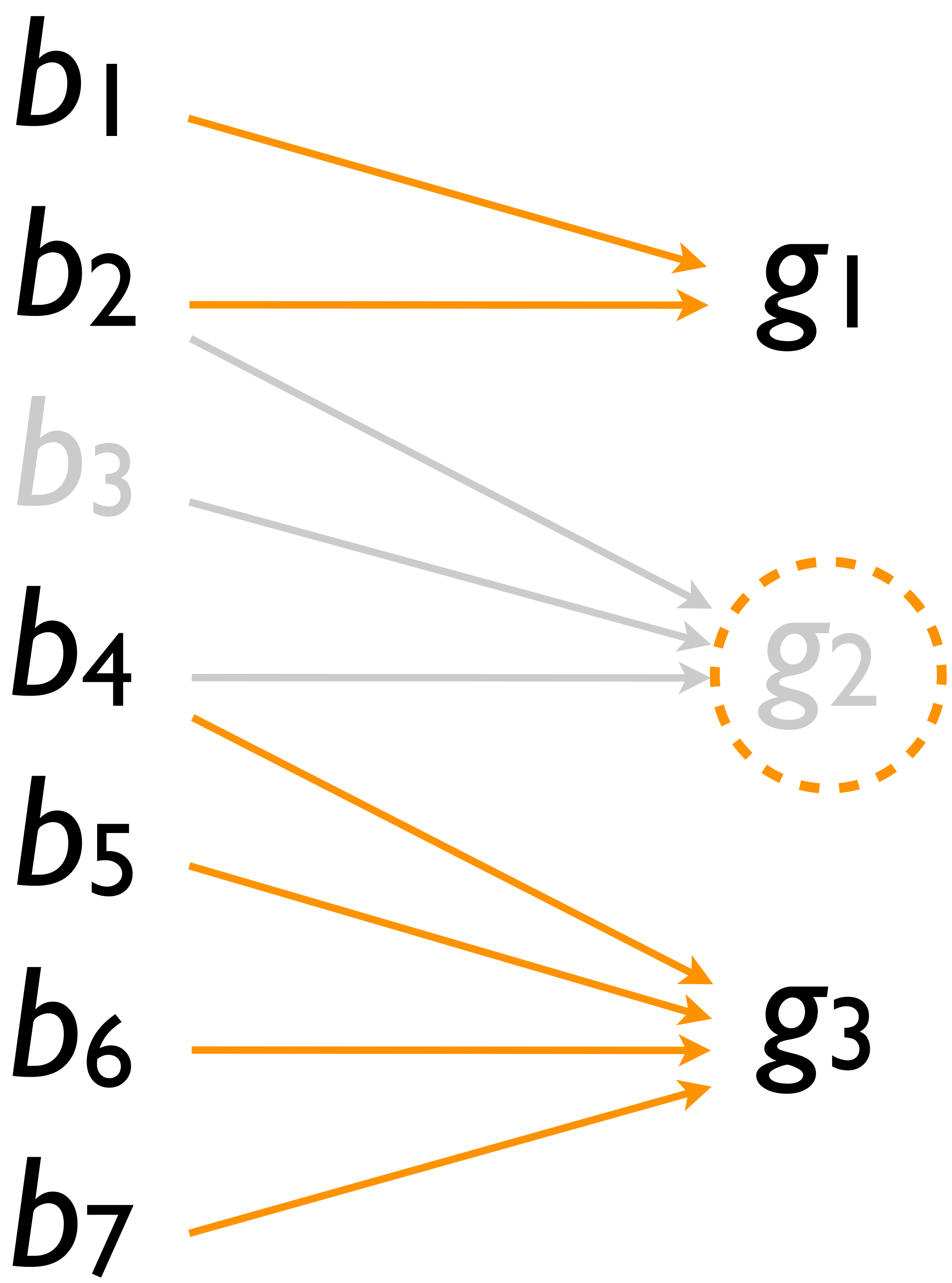
  (In this case, $|(d(R))^{-1}(g_2)| = |\{b_3\}| = 1 \leq \lfloor 7/3 \rfloor = 2$.)

R



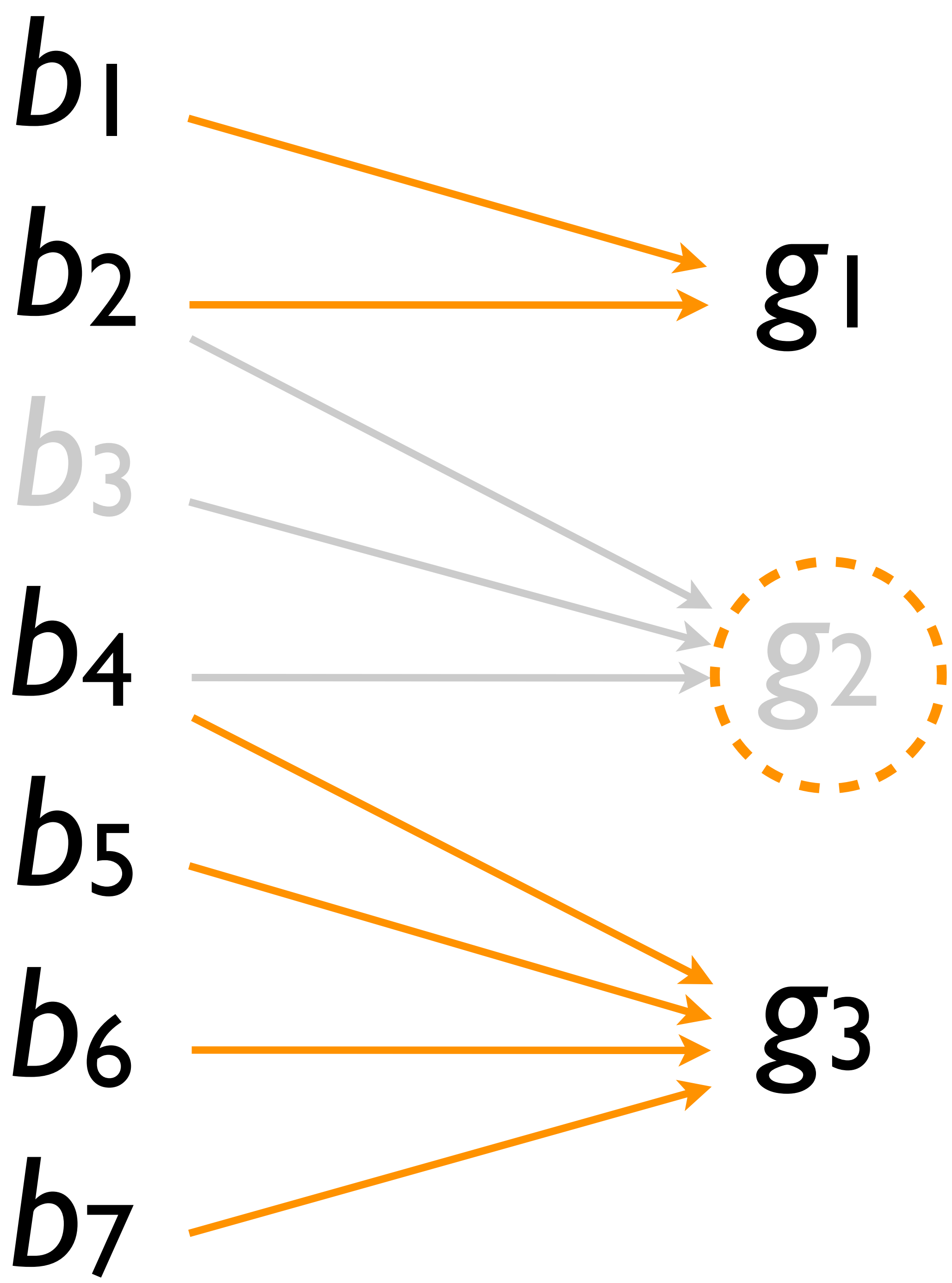
- Think of the relation R' obtained from R by removing that element g (= g_2 in this case).

R'



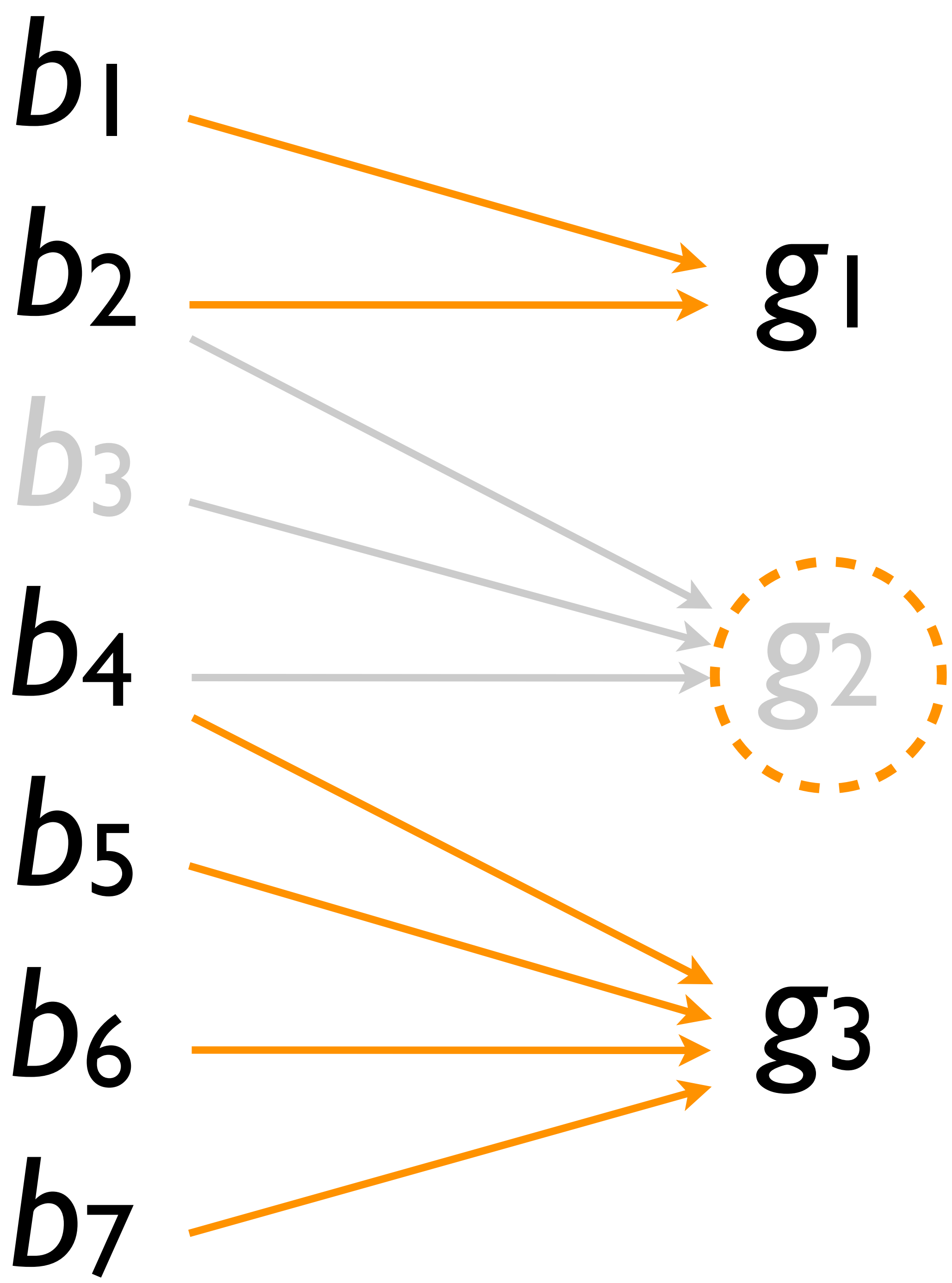
- Think of the relation R' obtained from R by removing that element g (= g_2 in this case).

R'



- In R', the elements of $(d(R))^{-1}(g)$ have also been removed.

R'



- Since $|(d(R))^{-1}(g)| \leq \lfloor m/n \rfloor$, we have $R'(m', n-1)$ and hence $R(m', n-1)$ for some $m' \geq m - \lfloor m/n \rfloor$.

- Since $m \geq n$,

$$\begin{aligned} m' &\geq m - \lfloor m/n \rfloor \\ &\geq n \cdot \lfloor m/n \rfloor - \lfloor m/n \rfloor \\ &\geq (n-1) \cdot \lfloor m/n \rfloor \\ &\geq n - 1. \end{aligned}$$

- So Lemma 11 implies

$$R(m', n-1) \models R(m - \lfloor m/n \rfloor, n-1).$$

- Hence $R(m - \lfloor m/n \rfloor, n-1)$. QED.

Inference Rules

- We write $R(m, n) \vdash R(k, l)$ iff $R(k, l)$ can be deduced from $R(m, n)$ by the following rules of inference:

(R2-1)

$$\frac{\mathrm{R}(m, n) \quad m > 1}{\mathrm{R}(m-1,\ n-\lfloor n/m \rfloor)}$$

(R2-2)

$$\frac{\mathrm{R}(m, n) \quad n > 1}{\mathrm{R}(m-\lfloor m/n \rfloor,\ n-1)}$$

Inference Rules

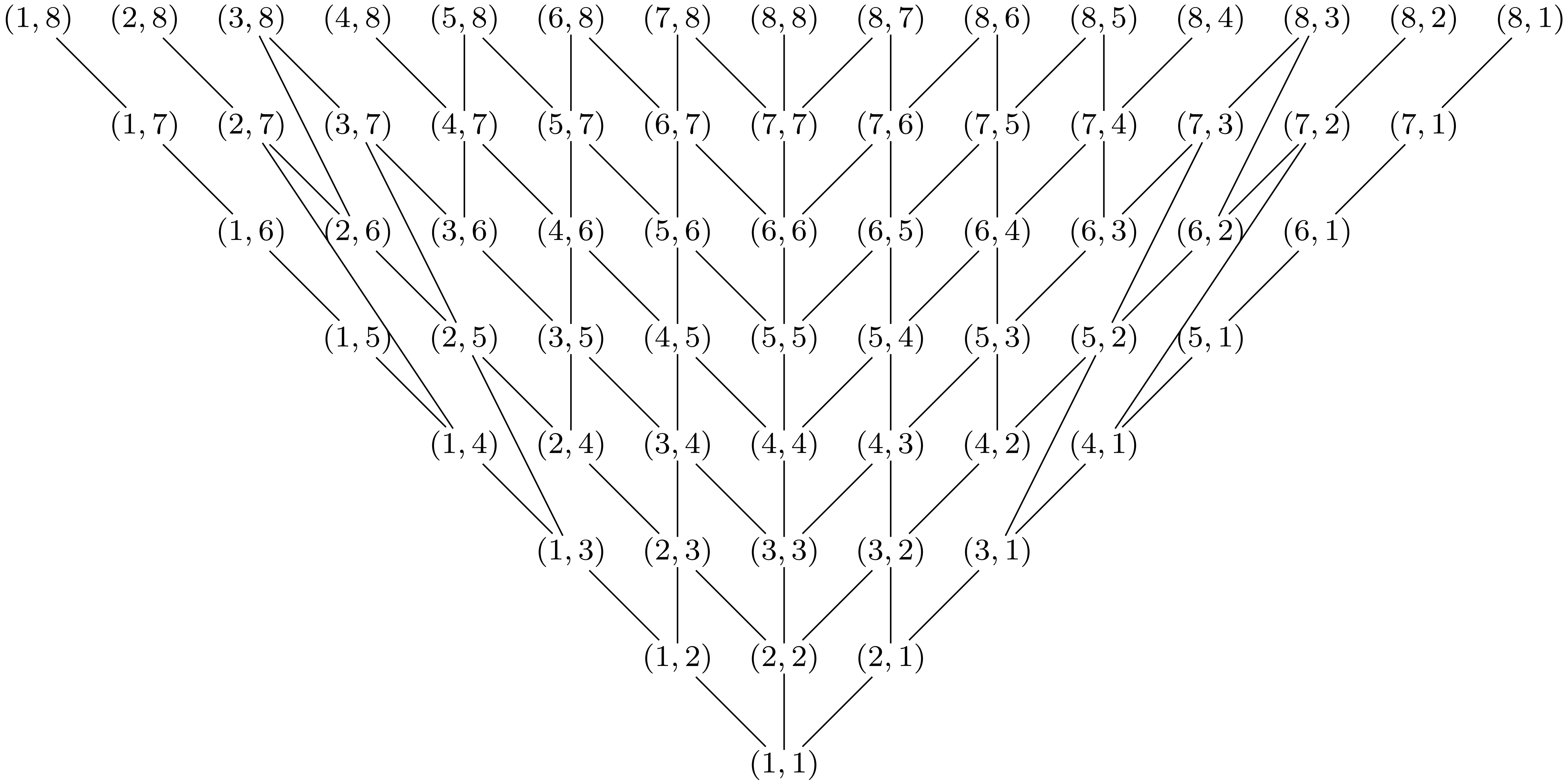
Theorem (soundness and completeness).

$$R(m, n) \vdash R(k, l) \text{ iff } R(m, n) \models R(k, l).$$

- When $m \geq n$, the entailment relation is characterized by

$$k \leq m \wedge l \leq n \wedge$$
$$l \leq k \leq l \cdot \lfloor m/n \rfloor + \min(m \bmod n, l).$$

Inference Rules



A Hasse diagram of entailment

Ternary Cumulative Sentences

(R3-1)

$$\frac{R(m, n, p) \quad m > n \quad m > n+p-2}{R(m-1, n, p)}$$

(R3-2)

$$\frac{R(m, n, p) \quad 2(n-p+1) \geq m > p \quad 2(m-p+1) \geq n > p \quad p \geq 2}{R(m-1, n-1, p)}$$

Ternary Cumulative Sentences

(R3-3)

$$\frac{R(m, n, p) \quad m - \lfloor (m-1)/(n-p+1) \rfloor \geq n+p-3 \quad n > p \geq 2}{R(m - \lfloor (m-1)/(n-p+1) \rfloor, n-1, p)}$$

Concluding Remarks

- To be able to evaluate claims about scalar implicatures about cumulative sentences, it is important to know when one such sentence entails another such sentence.

Concluding Remarks

- For the entailment relation between binary cumulative sentences, we obtained
  - a complete axiomatization with two inference rules.
  - a characterization of this relation.

Concluding Remarks

- For the entailment relation between ternary cumulative sentences,
  - we found some valid inference rules.
  - we hope to formulate a complete axiomatization in the future.

Concluding Remarks

- Ultimately, we hope to find general results for *k*-ary cumulative sentences.

THANK YOU!