Toward a Logic of Cumulative Quantification

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The notion of cumulative quantification was first introduced by Remko Scha. Distinct from subject-wide-scope and object-wide-scope readings.
Cumulation in Generalized Quantifier Theory

600 Dutch firms use 5000 American computers. Three boys kissed five girls.

\[ \text{Cum}(Q_1, Q_2) \, R \iff Q_1x \, \exists y \, R(x, y) \land Q_2y \, \exists x \, R(x, y) \]

\[ \llbracket n \, N \rrbracket = \{ Y \mid |\llbracket N \rrbracket \cap Y| = n \} \]

Cum takes two type <1> quantifiers and returns a type <2> quantifier.
Meanings of Numerals

John kissed five girls

\[
|\{ \text{girl} \} \cap \{ x \mid \exists x \text{ kissed}(\text{John}, x) \}| = 5
\]

\[
|\{ \text{girl} \} \cap \{ x \mid \exists x \text{ kissed}(\text{John}, x) \}| \geq 5
\]

“John kissed five girls” is consistent with “John kissed more than five girls”.

Scalar Implicatures

John kissed seven girls
John kissed six girls
John kissed five girls
John kissed four girls

“John kissed five girls” implicates “¬(John kissed six girls)”

Stronger alternatives to the utterance are negated.
Meanings of Numerals

John kissed five girls

\[ |\mathbb{[\text{girl}] \cap \{ x \mid \mathbb{[\text{kissed}] (\text{John, } x) \} }| = 5 \]

\[ |\mathbb{[\text{girl}] \cap \{ x \mid \mathbb{[\text{kissed}] (\text{John, } x) \} }| \geq 5 \]

\[ \mathbb{[n N]} = \{ Y \mid |\mathbb{[N]} \cap Y| \geq n \} \]

\[ \exists X \subseteq \mathbb{[\text{girl}]}(|X| = 5 \land \mathbf{**[\text{kissed}] (\text{John, } X)}) \]

\[ \mathbb{[n N]} = \{ Y \mid \exists X \subseteq \mathbb{[N]}(|X| = n \land X \in Y) \} \]
Meanings of Cumulative Sentences

\[
\llbracket m \mathbb{N}_1 \lor n \mathbb{N}_2 \rrbracket \iff \\
\exists X \subseteq \llbracket \mathbb{N}_1 \rrbracket (|X| = m \land \exists Y \subseteq \llbracket \mathbb{N}_2 \rrbracket (|Y| = n \land \star\star \llbracket \mathbb{V} \rrbracket (X, Y)))
\]

\[\star\star R(X, Y) \iff \forall x \in X \exists y \in Y R(x, y) \land \forall y \in Y \exists x \in X R(x, y)
\iff \exists R' \subseteq R (X = \text{dom}(R') \land Y = \text{ran}(R'))
\]

Krifka 1999, Landman 2000

\[\star\star R\text{ is the “cumulative closure” of } R.\]
Meanings and Implicatures(?)

Krifka-Landman semantics amounts to:

\[
\begin{align*}
\llbracket m \ N_1 \ V \ n \ N_2 \rrbracket & \iff \exists R' \subseteq R (|\text{dom}(R')| = m \land |\text{ran}(R')| = n) \\
\text{where } R &= \llbracket V \rrbracket \cap (\llbracket N_1 \rrbracket \times \llbracket N_2 \rrbracket)
\end{align*}
\]

which is weaker than Scha’s truth conditions:

\[
|\text{dom}(R)| = m \land |\text{ran}(R)| = n
\]
Pragmatic Scale for Cumulative Sentences

six
five
four
three boys kissed

two
one

eight
seven
six
five girls

four
three
Abbreviation: Write $R(m, n)$ for

$$\exists R' \subseteq R(|\text{dom}(R')| = m \land |\text{ran}(R')| = n)$$

What is said:

$$[m \mathbb{N}_1 \lor n \mathbb{N}_2] \iff R(m, n)$$

where $R = [V] \cap ([\mathbb{N}_1] \times [\mathbb{N}_2])$

Krifka-Landman implicature:

$$\forall m' \forall n'(R(m', n') \rightarrow (m' < m \lor n' < n \lor (m' = m \land n' = n)))$$

or, equivalently,

$$\forall m' \forall n'(R(m', n') \rightarrow (m' \leq m \land n' \leq n))$$

The idea is that if $m' \geq m \land n' \geq n \land \neg (m' = m \land n' = n)$, then $R(m', n')$ is “higher up” in the scale than $R(m, n)$, so the assertion of $R(m, n)$ implicates $\neg R(m', n')$. By taking contrapositive, we get $R(m', n') \rightarrow m' < m \lor n' < n \lor (m' = m \land n' = n)$, which is the first formulation of the Krifka–Landman implicature.

To see that the first formulation implies the second, suppose $R(m, n) \land R(m', n')$. Then $\neg R(X,Y)$ and $\neg R(X',Y')$ with $|X| = m$, $|Y| = n$, $|X'| = m'$, $|Y'| = n'$. From this we get $\neg R(X \cup X',Y \cup Y')$, so $R(m'', n'')$ with $m'' \geq \max(m,m')$ and $n'' \geq \max(n,n')$. If $m' > m$, then $m'' > m$ and $n'' \geq n$, so $\neg R(m'', n'')$ should be an implicature, contradicting $R(m'', n'')$. So we must have $m' \leq m$. Similarly, we can derive $n' \leq n$.

This shows that in the presence of $R(m, n)$, the two formulations of the implicature are equivalent.
Three boys kissed five girls

To be maximally informative, should say “Four boys kissed five girls”.

\[ b_1 \rightarrow g_1 \\
 b_2 \rightarrow g_2 \\
 b_3 \rightarrow g_3 \\
 b_4 \rightarrow g_4 \rightarrow g_5 \]
Four boys kissed five girls

$\begin{align*}
&b_1 \rightarrow g_1 \\
&b_2 \rightarrow g_2 \\
&b_3 \rightarrow g_3 \\
&b_4 \rightarrow g_4 \\
&\quad \quad \rightarrow g_5
\end{align*}$

four boys kissed five girls $\not\equiv$ three boys kissed five girls
When does $R(m, n)$ logically imply $R(k, l)$?

$$R(m, n) \iff \exists R' \subseteq R(|\text{dom}(R')| = m \land |\text{ran}(R')| = n)$$

$$m \in N_1 \lor n \in N_2 \models k \in N_1 \lor l \in N_2$$

$$R(m, n) \models R(k, l)$$

When does $R(m, n)$ logically imply $R(k, l)$?
Deterministic Reducts

• If \( R \) is a binary relation, its deterministic reduct is:

\[
d(R) = \{ (x, y) \in R \mid \forall y' ((x, y') \in R \rightarrow y' = y) \}.
\]

• \((x, y) \in d(R)\) means that \( y \) is the only element related to \( x \) by \( R \).

• \( d(R) \) is a partial function.
Deterministic Reducts

\[ b_1 \rightarrow g_1 \]
\[ b_2 \rightarrow g_2 \]
\[ b_3 \rightarrow g_3 \]
\[ b_4 \rightarrow g_4 \]
\[ b_5 \]
Deterministic Reducts

\[ b_1 \rightarrow g_1 \]
\[ b_2 \rightarrow g_2 \]
\[ b_3 \rightarrow g_3 \]
\[ b_4 \rightarrow g_4 \]
\[ b_5 \rightarrow g_4 \]
Deterministic Reducts

• \((d(R))^{-1}\) is the inverse image of \(d(R)\):

\[
(d(R))^{-1}(y) = \{ x \mid (x, y) \in d(R) \}.
\]
$d(R)$

$b_1 \rightarrow g_1$

$b_2 \rightarrow g_2$

$b_3 \rightarrow g_3$

$b_4 \rightarrow g_4$

$b_5 \rightarrow g_4$
$$(d(R))^{-1}$$

$$(d(R))^{-1}(g_1) = \{b_1\}$$

$$(d(R))^{-1}(g_2) = \emptyset$$

$$(d(R))^{-1}(g_3) = \emptyset$$

$$(d(R))^{-1}(g_4) = \{b_4, b_5\}$$
Entailments

Lemma 11

If $m > n$, then $R(m, n) \models R(m-1, n)$. 

Suppose $B = \text{dom}(R)$, $G = \text{ran}(R)$, $|B| = m$, $|G| = n$, so $R(m, n)$. 
• Since $d(R^{-1})$ is a partial function from $G$ to $B$ and since $|B| > |G|$...
\[
(d(R^{-1}))^{-1}
\]

...there must be some \( b \in B \) s.t.
\[
(d(R^{-1}))^{-1}(b) = \emptyset.
\]

(Here, we have \( (d(R^{-1}))^{-1}(b_3) = \emptyset \) for instance.)
Think of the relation $R'$ obtained from $R$ by removing that element $b$ (= $b_3$ in this case).
Since $R'(m-1, n)$, $R(m-1, n)$. QED.
Entailments

Lemma 12

If \( m \geq n > 1 \), then \( R(m, n) \models R(m-\lfloor m/n \rfloor, n-1) \).

Here, \( \lfloor m/n \rfloor \) denotes the quotient of \( m \) divided by \( n \) (e.g., \( \lfloor 12/5 \rfloor = 2 \)).
• Suppose
$B = \text{dom}(R)$,
$G = \text{ran}(R)$,
$|B| = m$,
$|G| = n$,
so $R(m, n)$. 
$d(R)$

$\begin{align*}
&b_1 \\
&b_2 \\
&b_3 \\
&b_4 \\
&b_5 \\
&b_6 \\
&b_7
\end{align*}$
There must be some $g \in G$ s.t.

\[ |(d(R))^{-1}(g)| \leq \left\lfloor \frac{m}{n} \right\rfloor. \]

(In this case, \( |(d(R))^{-1}(g_2)| = |\{b_3\}| = 1 \leq \left\lfloor \frac{7}{3} \right\rfloor = 2 \).
• Think of the relation $R'$ obtained from $R$ by removing that element $g$ (= $g_2$ in this case).
Think of the relation $R'$ obtained from $R$ by removing that element $g$ (= $g_2$ in this case).
In $R'$, the elements of $(d(R))^{-1}(g)$ have also been removed.
Since \(|(d(R))^{-1}(g)| \leq \lfloor m/n \rfloor\), we have \(R'(m', n-1)\) and hence \(R(m', n-1)\) for some \(m' \geq m-\lfloor m/n \rfloor\).
• Since \( m \geq n \),
\[
m' \geq m - \lfloor m/n \rfloor
\geq n \cdot \lfloor m/n \rfloor - \lfloor m/n \rfloor
\geq (n - 1) \cdot \lfloor m/n \rfloor
\geq n - 1.
\]

• So Lemma 11 implies
\[
R(m', n-1) \models R(m-\lfloor m/n \rfloor, n-1).
\]

• Hence \( R(m-\lfloor m/n \rfloor, n-1) \). QED.
Inference Rules

- We write $R(m, n) \vdash R(k, l)$ iff $R(k, l)$ can be deduced from $R(m, n)$ by the following rules of inference:

(R2-1)  
\[
\frac{R(m, n) \quad m > 1}{R(m-1, n - \lfloor n/m \rfloor)}
\]  

(R2-2)  
\[
\frac{R(m, n) \quad n > 1}{R(m - \lceil m/n \rceil, n-1)}
\]
Inference Rules

Theorem (soundness and completeness).

\[ R(m, n) \vdash R(k, l) \text{ iff } R(m, n) \models R(k, l). \]

- When \( m \geq n \), the entailment relation is characterized by

\[
\begin{align*}
  k &\leq m \land l \leq n \land \\
  l &\leq k \leq l \cdot \lfloor m/n \rfloor + \min(m \mod n, l).
\end{align*}
\]
Inference Rules

A Hasse diagram of entailment
Ternary Cumulative Sentences

(R3-1)
\[ R(m, n, p) \quad m > n \quad m > n+p-2 \]
\[ \therefore R(m-1, n, p) \]

(R3-2)
\[ R(m, n, p) \quad 2(n-p+1) \geq m > p \]
\[ 2(m-p+1) \geq n > p \quad p \geq 2 \]
\[ \therefore R(m-1, n-1, p) \]
Ternary Cumulative Sentences

\[(R3-3)\]
\[R(m, n, p) \quad m - \left\lfloor \frac{m-1}{n-p+1} \right\rfloor \geq n+p-3\]
\[n > p \geq 2\]
\[
\frac{R(m - \left\lfloor \frac{m-1}{n-p+1} \right\rfloor, n-1, p)}{}
\]
Concluding Remarks

• To be able to evaluate claims about scalar implicatures about cumulative sentences, it is important to know when one such sentence entails another such sentence.
Concluding Remarks

• For the entailment relation between binary cumulative sentences, we obtained
  - a complete axiomatization with two inference rules.
  - a characterization of this relation.
Concluding Remarks

• For the entailment relation between ternary cumulative sentences,
  - we found some valid inference rules.
  - we hope to formulate a complete axiomatization in the future.
Concluding Remarks

- Ultimately, we hope to find general results for $k$-ary cumulative sentences.
THANK YOU!