Boundary Aligned Smooth 3D Cross-Frame Field

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Background: 2D Frame Field & Quad Meshing

- 2D Frame Field
  - Auto-computed

- UV Parameterization
  - Auto-computed

Mixed-Integer Quadrangulation [Bommes,Zimmer,Kobbelt,SIGGRAPH09]
Background: 3D Frame Field & Hex Meshing

- 3D Frame Field ➔ Heuristic
- UVW Parameterization ➔ Auto-computed

“Meta-Mesh” to define 3D Frame Field

UVW Parameterization = Hex Mesh
Definition of 3D Frame

• Don’t care about orientation / ordering of axes

\[ R_1 = R_2 = \cdots = R_{24} \]

• Question: How distant is \( R_a \) from \( R_b \)?

• Key insight: \( h(s) := s_x^2 s_y^2 + s_y^2 s_z^2 + s_z^2 s_x^2, \quad s \in S^2 \)

• Invariant under sign flip / axis reordering!
Distance between 3D Frames

\[ h(s) := s_x^2 s_y^2 + s_y^2 s_z^2 + s_z^2 s_x^2 \]

\[ d(R_a, R_b) := \int_{s \in S^2} \left( h(R_a^T s) - h(R_b^T s) \right)^2 ds \]

- Integral over an entire sphere ➔ Spherical Harmonics!
Basics of Spherical Harmonics

• Something like Fourier series on sphere

\[ f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{f}_l^m Y_l^m(\theta, \phi) \]

• Orthonormality:

\[ \int_{s \in S^2} Y_{l_1}^{m_1}(s) Y_{l_2}^{m_2}(s) \, ds = \begin{cases} 1 & \text{if } m_1 = m_2 \\ 0 & \text{otherwise} \end{cases} \]
Frame represented by SH

\[ h(s) = -\frac{2\sqrt{\pi}}{15} \left( Y_4^0(s) + \sqrt{\frac{5}{7}} Y_4^4(s) + 16\sqrt{\pi} Y_0^0(s) \right) \]

- “Frequency” unaffected by rotation:
- Frame represented as SH coeffs for band \( l = 4 \) (i.e. 9-vector):
- Coeffs mapped by some 9x9 matrix \( \hat{R} \):
- Distance between \( R_a \) & \( R_b \):

\[ \hat{f}[R] = \hat{R} \hat{h} \]

\[ d(R_a, R_b) = \| \hat{R}_a \hat{h} - \hat{R}_b \hat{h} \|^2 \]
Computing 9x9 $\hat{R}$ from 3D rotation $R$

• Not immediately obvious

• Insight: Obvious for certain cases: $R_Z^\theta$ & $R_X^{\pi/2}$
  • (Not sure...)

➔ Represent rotation by ZYZ Euler angle

  • $R(\alpha, \beta, \gamma) := R^\gamma_Z \left( R^\beta_Y \right) R^\alpha_Z = R^\gamma_Z \left( R_{X}^{-\frac{\pi}{2}} R_Z^\beta R_{X}^{\frac{\pi}{2}} \right) R^\alpha_Z$

  • $\hat{R}(\alpha, \beta, \gamma) = \hat{R}^\gamma_Z \left( \hat{R}_{X}^{-\frac{\pi}{2}} \hat{R}_{Z}^\beta \hat{R}_{X}^{\frac{\pi}{2}} \right) \hat{R}_{Z}^\alpha$
Frame that aligns with boundary surface

• Frame $[R]$ aligns with:
  • Z axis iff $\hat{f}_{[R]}(0) = \sqrt{7}$ (Proof in Appendix)
  • Surface normal $n$ iff $(\hat{R}_{n\rightarrow Z} \hat{f}_{[R]})(0) = \sqrt{7}$

• $R_{n\rightarrow Z}$: Rotation that brings $n$ to Z axis
  $\alpha = -\text{atan2}(n_y, n_x), \quad \beta = -\text{acos}(n_z), \quad \gamma = 0$
Discretization & Objective

• Tetrahedral mesh over domain $\Omega$

• Frame var $\hat{f}_{p_i}$ at center $p_i$ of every (interior/exterior) triangle TRI$_i$

• Piecewise-linear frame field $\hat{f}$
  • Gradient $\nabla \hat{f}_{TET_j}$ constant within each tetrahedron TET$_j$

• Objective to be minimized:

$$E_{\text{smooth}} := \sum_{TET_j} \text{volume}(TET_j) \sum_{m=-4}^{4} \| \nabla \hat{f}_{TET_j}(m) \|^2$$

$$E_{\text{align}} := \sum_{\text{TRI}_i \in \partial \Omega} \text{area}(\text{TRI}_i) \| (\hat{R}_{n_i} \cdot z \hat{f}_{p_i})(0) - \sqrt{7} \|^2$$

$$E_{\text{full}} := \frac{E_{\text{smooth}}}{\text{volume}(\Omega)^{1/3}} + w_{\text{align}} \frac{E_{\text{align}}}{\text{area}(\partial \Omega)}$$
Optimization

• Energy quadratic in \( \{ \hat{f}_i \} \) \( \Rightarrow \) Simple Laplace-like least squares \(<\text{Step 1}>\)

• Problem: Arbitrary \( \hat{f}_i \) doesn’t represent rotation!

\( \Rightarrow \) \(<\text{Step 2}>\) Project \( \hat{f}_i \) to its closest rotation \( R(\alpha_i, \beta_i, \gamma_i) \)
  • (Not sure how to do it...)

• \(<\text{Step 3}>\) Using \( \Phi_i := (\alpha_i, \beta_i, \gamma_i) \) as initial guess, run \textit{nonlinear optimization} over \( \{ \Phi_i \} \)
  • L-BFGS (solver: ALGLIB, dlib, etc)
  • (Not sure about analytic form of derivative...)

\[
\hat{f}_0 \leftarrow \arg \min_f E_f(\hat{f}) \\
\text{for all rotation } \Phi_i = (\alpha_i, \beta_i, \gamma_i) \text{ do} \\
\Phi_{0,i} \leftarrow \arg \min_{\Phi_i} \| \hat{f}_{0,i} - \hat{R}(\Phi_i) \hat{h} \|^2 \\
\text{end for} \\
\text{repeat} \\
\text{L-BFGS iteration for } \arg \min_{\Phi} E_f(\hat{f}[\hat{R}(\Phi)]) \\
\text{until } -\frac{\Delta E_f}{E_f} < 10^{-5}
\]
Results & Performance

<table>
<thead>
<tr>
<th>Model</th>
<th>Tetrahedron</th>
<th>Memory</th>
<th>Iteration</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere</td>
<td>194k</td>
<td>1.1G</td>
<td>838</td>
<td>21.8m</td>
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<td>Fan disk</td>
<td>301k</td>
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<td>507k</td>
<td>3.0G</td>
<td>667</td>
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<td>Rock arm</td>
<td>947k</td>
<td>5.8G</td>
<td>928</td>
<td>155.9m</td>
</tr>
</tbody>
</table>
Questions

• Regarding implementation:
  • Expressions for $\hat{R}_Z^\theta$ & $\hat{R}_X^{\pi/2}$
  • Projection of $\hat{f}_i$ to its closest rotation $R(\alpha_i, \beta_i, \gamma_i)$
  • Analytic derivative of $E_{\text{full}}$ w.r.t. $\{\Phi_i\}$

• Possible idea for improvement:
  • Can we sidestep nonlinear optimization by alternating Laplace smoothing and “normalization”?