Matroidal Choice Functions

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Abstract

In some game-theoretic models, an agent is supposed to choose a subset of items under a matroid constraint. Recently, it was pointed out that a choice rule defined by the standard greedy algorithm for matroids fulfills the "substitutability," which guarantees the existence of an equilibrium in two-sided market models.

In this paper, we introduce a notion of "matroidal choice functions" to capture the entire class of substitutable choice rules under a matroid constraint. For such functions, we provide two characterizations: the one is by the behaviour of an online greedy algorithm and the other is by a local condition. We also show that matroidal choice functions are closely related to valuated matroids and discrete concave functions.

Keywords: Matroid, Choice function, Substitutability, Greedy algorithm, Valuated matroid, Stable matching

1 Introduction

In some game-theoretic models, an agent chooses a subset of offered items under a matroid constraint. For example, in the college admission model of Gale and Shapley [9], a college takes applicants up to its quota $q \in \mathbf{N}$, i.e., it has a uniform matroid constraint of rank q. Also, a laminar matroid naturally arises if a college has a nested structure [7], and a transversal matroid arises if a college considers an assignment of students to roles.

Under a matroid constraint, if we have a total order on individual applicants, the most naive way of choosing a subset of applicants is to execute the standard greedy algorithm, i.e., we pick up available applicants from the highest to the lowest preserving the matroid constraint. Fleiner [5] showed that such a choice rule fulfills the "substitutability" ("comonotonicity," in his terminology), an essential property for the existence of an equilibrium in two-sided market models [20].

There are, however, more general substitutable choice rules under a matroid constraint as follows. Assume that a college admits at most two applicants. The applicants are men m_1, m_2 and women w_1, w_2. The college prefers m_1 to m_2 and w_1 to w_2, and admits members of both sexes as equally as possible. Hence, its first choice is m_1w_1. However if the available set is $m_2w_1w_2$ (resp., $m_1m_2w_2$), then the choice is m_2w_1 (resp., m_1w_2). In the case of $m_2w_1w_2$, man m_2 is preferred to woman w_2, in contrast to the case of $m_1m_2w_2$, in which w_2 is preferred to m_2. Thus we cannot define a total order on individual applicants while this choice rule fulfills the substitutability.

The aim of this paper is to analyze the nature of substitutable choice rules under a matroid constraint. We express choice rules by choice functions, and introduce a notion of "matroidal choice functions" as the whole class of substitutable choice functions under a matroid constraint. We show that each matroidal choice function can be represented by a "de-cycle function," which is a function on the circuit family of a matroid indicating one element of each circuit. We show that there is a one-to-one correspondence between matroidal choice functions and de-cycle functions satisfying the "coherency," and we characterize the coherent de-cycle functions in the following two ways.

The first characterization is by an online greedy algorithm. Assume that a de-cycle function indicates the worst element of each circuit, and we are given a subset of the ground set as a sequence of elements. Then, we can naturally conceive the following algorithm: Start with the empty set and add elements of the sequence, but whenever a circuit comes up, eliminate the element indicated by the de-cycle function. This algorithm works well for some kind of de-cycle functions. For example, if there are positive weights on elements and the de-cycle function indicates the minimum weight element in each circuit, the output is the maximum weight independent subset regardless of the order of the sequence. For general de-cycle functions, however, the output of the algorithm differs depending on the order of the sequence. We show that the output of the algorithm is independent of the order if and only if the de-cycle function is coherent. In addition, the output of the algorithm with a coherent de-cycle function is just a subset returned by the corresponding matroidal choice function. Thus, matroidal choice functions are characterized as choice functions computable by the above online greedy algorithm with coherent de-cycle functions.

The other characterization of the coherency is based on a local condition. We introduce the concept of "minimal pair of circuits," which means a pair of distinct circuits whose union is inclusion-wise minimal among all such unions. While the original definition of the coherency requires a particular condition for all subsets of the ground set, the local characterization is concerned only with minimal pairs of circuits.

Matroidal choice functions are closely related to nonlinear generalizations of weighted matroids, such as valuated matroids and $\mathrm{M}^\natural$-concave functions. A valuated matroid, introduced by Dress and Wenzel [2, 4], is a matroid equipped with a function defined on the base family satisfying a quantitative version of the exchange axiom. For this structure, various efficient algorithms of weighted matroids have been extended: the greedy algorithm and the valuated matroid intersection algorithm [12, 13]. For valuated matroids, there are equivalent sets of axioms in terms of a function on the independent sets [14] and in terms of vectors on the circuits [17]. Generalizing the idea of valuated matroid, Murota and Shioura [16] defined the $\mathrm{M}^\natural$-concavity for functions on the integer lattice. These structures are called discrete concave functions [15].

Our online greedy algorithm mentioned above shows a marked similarity to the maximization algorithm for valuated matroids. Also, it is known that a choice rule which maximizes the value of an $\mathrm{M}^\natural$-concave function fulfills the substitutability [8, 18]. As suggested by these facts, there is a close relationship between matroidal choice functions and discrete concave functions. We show that a choice function defined by maximizers of an $\mathrm{M}^\natural$-concave set function is a matroidal choice function under certain conditions. It is also shown that not all matroidal choice functions arise from discrete concave functions. Thus, the notion of matroidal choice functions can be regarded as an abstraction of combinatorial aspects of maximization algorithms of discrete concave functions.

The rest of this paper is organized as follows. Sections 2 provides preliminaries on matroids. Matroidal choice functions are introduced in Section 3 and characterized in Sections 4 and 5 by an online greedy algorithm and by a local condition, respectively. Finally, Section 6 explains how matroidal choice functions relate to valuated matroids and $\mathrm{M}^\natural$-concave functions.

2 Matroids

This section provides preliminaries on matroids. See Oxley [19] for more information. For a subset X and an element e of a finite set E, we denote $X \cup \{e\}$ by $X + e$ and $X \setminus \{e\}$ by $X - e$.

For a finite set E and a family $\mathcal{I} \subseteq 2^E$, a pair $\mathbf{M} = (E, \mathcal{I})$ is called a *matroid* if it satisfies the following (I1)–(I3):

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I_1 \subseteq I_2 \in \mathcal{I}$, then $I_1 \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there exists an element $e \in I_2 \setminus I_1$ such that $I_1 + e \in \mathcal{I}$.

We call the family $\mathcal{I}$ the *independent set family* of $\mathbf{M}$. We say that a set $X \subseteq E$ is independent if $X \in \mathcal{I}$. Otherwise, we say that X is dependent.

For a set $X \subseteq E$, a maximal independent subset of X is called a *base* of X. We denote by $\mathcal{B}(X)$ the set of bases of X, i.e.,

$$\mathcal{B}(X) = \{\, B \subseteq X \mid B \in \mathcal{I},\ B + e \notin \mathcal{I}\ (\forall e \in X \setminus B) \,\}.$$

In particular, we write $\mathcal{B}$ for $\mathcal{B}(E)$, which is called the base family of $\mathbf{M}$.

The *circuit family* $\mathcal{C} \subseteq 2^E$ of $\mathbf{M}$ is defined by

$$\mathcal{C} = \{\, C \subseteq E \mid C \notin \mathcal{I},\ C - e \in \mathcal{I}\ (\forall e \in C) \,\},$$

and each member is called a circuit. That is, a circuit is a minimal dependent set. It is known that $\mathcal{C}$ satisfies the following (C1)–(C3):

(C1) $\emptyset \notin \mathcal{C}$.

(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \neq C_2$, then for all $e \in C_1 \cap C_2$ there exists $C \in \mathcal{C}$ such that $C \subseteq (C_1 \cup C_2) - e$.

The *rank function* of $\mathbf{M}$ is a function $r : 2^E \to \mathbf{Z}$ defined by

$$r(X) = \max\{\ |I| : I \in \mathcal{I},\quad I \subseteq X\} \quad (X \subseteq E).$$

It is known that r satisfies the following (R1)–(R3):

(R1) $\forall X \subseteq E, \quad 0 \leq r(X) \leq |X|$.

(R2) If $X \subseteq Y$, then $r(X) \leq r(Y)$.

(R3) $\forall X, Y \subseteq E, \quad r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$.

We call $r(E)$ the rank of $\mathbf{M}$. In this paper, we often use the following fact.

Fact 2.1. For any $X \subseteq E$, every $B \in \mathcal{B}(X)$ satisfies $|B| = r(X)$. ■

3 Matroidal Choice Functions

In this section, we introduce the concept of "matroidal choice function" and provide a representation method with circuit families.

3.1 Definition

A *choice function* on a finite set E is a function $F : 2^E \to 2^E$ such that $F(X) \subseteq X$ for any $X \subseteq E$. We interpret $F(X)$ as the most preferred subset of X. A choice function F is said to be *substitutable* if it satisfies

(Sub) $X \subseteq Y \implies F(Y) \cap X \subseteq F(X)$.

One can easily confirm that (Sub) is equivalent to

(Sub*) $X \subseteq Y \implies X \setminus F(X) \subseteq Y \setminus F(Y)$.

This says that an item rejected in some set will also be rejected if the set is expanded. We refer to both (Sub) and (Sub*) as the substitutability.

Let us define a "matroidal choice function" as a substitutable choice function which chooses one maximal subset under a matroid constraint.

Definition 3.1. For a matroid $\mathbf{M} = (E, \mathcal{I})$, a function $F : 2^E \to 2^E$ is said to be a *matroidal choice function* on $\mathbf{M}$ if it satisfies the substitutability and $F(X) \in \mathcal{B}(X)$ for each $X \subseteq E$.

We simply say F is a matroidal choice function if there is such a matroid $\mathbf{M}$, which is called the *underlying matroid* of F. ■

For any $X \subseteq E$, we see that $X \in \mathcal{I}$ implies $\mathcal{B}(X) = \{X\}$ and that $X \notin \mathcal{I}$ implies $X \notin \mathcal{B}(X)$. This leads to the following.

Proposition 3.2. For a matroidal choice function F on $\mathbf{M} = (E, \mathcal{I})$, a subset $X \subseteq E$ satisfies $F(X) = X$ if and only if $X \in \mathcal{I}$.

Given a matroidal choice function F, we can obtain its underlying matroid by setting $\mathcal{I} = \{\, X \subseteq E \mid F(X) = X \,\}$. ■

Since a matroidal choice function satisfies $F(X) \in \mathcal{B}(X)$, which implies $|F(X)| = r(X)$, the monotonicity of the rank function implies the following.

Proposition 3.3. A matroidal choice function F is *size-monotone*, i.e., $X \subseteq Y \subseteq E$ implies $|F(X)| \leq |F(Y)|$. ■

Remark 3.4. The substitutability is known as an essential condition for the existence of a stable matching in two-sided matching models [11, 20]. Moreover, combined with size-monotonicity (or the "law of aggregate demand"), it yields further results. For example, the set of stable matchings forms a distributive lattice [1, 5, 6], and the deferred acceptance algorithm is strategy-proof [10]. Since matroidal choice functions are substitutable and size-monotone, these results carry over to matching models with matroidal choice functions. ■

Here we give some examples of matroidal choice functions. We first provide the most basic example.

Example 3.5 (Standard Greedy Algorithm)**.** Let $\mathbf{M} = (E, \mathcal{I})$ be a matroid and $\succ$ be a total order on E such that $e_1 \succ e_2 \succ \cdots \succ e_n$, where $n = |E|$. Define a choice function $F : 2^E \to 2^E$ by letting $F(X)$ be the output of the following algorithm for every $X \subseteq E$: Let $F^0(X) := \emptyset$ and define $F^i(X)$ for each $i \in \{1, 2, \ldots, n\}$ by

$$F^i(X) := \begin{cases} F^{i-1}(X) + e_i & e_i \in X \text{ and } F^{i-1}(X) + e_i \in \mathcal{I}, \\ F^{i-1}(X) & \text{otherwise,} \end{cases} \tag{1}$$

and then let $F(X) := F^n(X)$. Then F is a matroidal choice function on $\mathbf{M}$. Indeed, we can observe that $F(X)$ is a base of X. Also, F is substitutable, which is shown by Fleiner [5, 6]. ■

Next, we provide two practical examples of matroidal choice functions. Each of them represents a criterion of selecting students of some college.

Example 3.6 (Nested Structure)**.** Assume that a college has a nested structure represented by a tree with node set V and root $r \in V$. Let $L \subseteq V$ be the set of *leaves*, i.e., nodes without children. Each leaf $l \in L$ has an ideal number p_l of applicants to accept. Then, p_v for each $v \in V \setminus L$ is defined by $p_v := \sum \{\, p_u \mid u \text{ is a child of } v \,\}$ inductively. Also, each $v \in V$ has a quota q_v with $q_v \geq p_v$. Let E_l be the set of possible applicants for each $l \in L$. Then, the set of possible applicants for $v \in V \setminus L$ is inductively defined by $E_v := \cup \{\, E_u \mid u \text{ is a child of } v \,\}$. In particular, we write $E := E_r = \cup \{\, E_l \mid l \in L \,\}$. Then, the family of acceptable applicant sets is

$$\mathcal{I} := \{\ Y \subseteq E : \ |Y \cap E_v| \leq q_v \ (\forall v \in V)\ \}.$$

The pair $(E, \mathcal{I})$ is a laminar matroid. The college also has a total order $\succ$ on E, where $e \succ e'$ means that e is better than e'. Assume that the applicants in $X \subseteq E$ apply for the college, i.e., for each l, the applicants in $X \cap E_l$ apply. Then the college decides whom to accept according to the following two steps.

Step 1. Each leaf takes applicants up to its ideal number: Let $F^0(X)$ be the direct sum of $\{\, F_l^0(X) \subseteq E_l \mid l \in L \,\}$, where each $F_l^0(X)$ is the set of top p_l elements of $X \cap E_l$ w.r.t. $\succ$ if $|X \cap E_l| > p_l$ and otherwise $X \cap E_l$. Note that $F^0(X) \in \mathcal{I}$ since $|F^0(X) \cap E_v| \leq p_v \leq q_v$ for every $v \in V$.

Step 2. The college additionally accepts applicants as far as quotas allow: Let $n = |X \setminus F^0(X)|$ and, for each $i \in \{1, 2, \ldots, n\}$, let e_i be the i-th element in $X \setminus F^0(X)$ w.r.t. $\succ$. Define $F^i(X)$ for each $i \in \{1, 2, \ldots, n\}$ by (1), and then let $F(X) := F^n(X)$.

We see that $F(X)$ is a base of X for the matroid $(E, \mathcal{I})$. Also, we can check that $F : 2^E \to 2^E$ satisfies the condition (Sub*). Thus, F is a matroidal choice function on $(E, \mathcal{I})$. ■

Example 3.7 (Role Assignment)**.** Let $G = (E, R; A)$ be a bipartite graph with node sets E, R and edge set A. Here, E represents the set of possible applicants for a college and so does R the set of roles to which accepted applicants are assigned. An arc $(e, r) \in A$ means that $e \in E$ can be assigned to $r \in R$. Then, the family of acceptable applicant sets is

$$\mathcal{I} := \{\, \partial M \cap E \mid M \subseteq A : \text{matching} \,\},$$

where $\partial M \subseteq E \cup R$ is the set of end vertices of $M \subseteq A$. The pair $(E, \mathcal{I})$ is a transversal matroid. We also have $w : A \to \mathbf{R}$, where $w(e, r)$ denotes a profit obtained by assigning e to r. We write $w(M) = \sum \{\, w(a) \mid a \in M \,\}$ for any $M \subseteq A$ and assume $w(M_1) \neq w(M_2)$ for any $M_1, M_2 \subseteq A$ with $M_1 \neq M_2$. Consider that the applicants in $X \subseteq E$ apply for the college. The college primarily wants to maximize the number of applicants to accept, and also wants to increase the total profit. Let $\mu_G(X)$ be the maximum size of a matching M with $\partial M \cap E \subseteq X$ and define $\mathcal{M}(X) \subseteq 2^A$ by

$$\mathcal{M}(X) := \{\, M \subseteq A \mid M : \text{matching},\ \partial M \cap E \subseteq X,\ |M| = \mu_G(X) \,\}.$$

Then, the chosen set $F(X) \subseteq X$ is defined as $F(X) := \partial M^* \cap E$, where M^* is the unique solution to $\max \{\, w(M) \mid M \in \mathcal{M}(X) \,\}$. By definition, we see that $F(X)$ is a base of X for the matroid $(E, \mathcal{I})$. Also, $F : 2^E \to 2^E$ satisfies the condition (Sub) as follows.

For $X \subseteq Y \subseteq E$, define $F(X)$ and $F(Y)$ as described above. Then, there are $M_X \in \mathcal{M}(X)$ and $M_Y \in \mathcal{M}(Y)$ such that

$$F(X) = \partial M_X \cap E, \quad w(M_X) > w(N)\ (\forall N \in \mathcal{M}(X) \setminus \{M_X\}),$$
$$F(Y) = \partial M_Y \cap E, \quad w(M_Y) > w(N)\ (\forall N \in \mathcal{M}(Y) \setminus \{M_Y\}).$$

Their symmetric difference $M_X \triangle M_Y$ is partitioned into alternating paths and cycles. Suppose, to the contrary, that there is $e \in F(Y) \cap X$ with $e \notin F(X)$. Then, $e \in \partial M_Y \setminus \partial M_X$, and hence e is the end vertex of some alternating path $P \subseteq M_X \triangle M_Y$ with $|P \cap M_Y| \geq |P \cap M_X|$. Let $N_X := M_X \triangle P$ and $N_Y := M_Y \triangle P$, which are matchings in G. In the case $|P|$ is odd, $|N_X| = |M_X| + 1$ and $\partial N_X \cap E = F(X) + e \subseteq X$, which contradicts $|M_X| = \mu_G(X)$. In the case $|P|$ is even, we have $\partial N_X \cap E = F(X)$ and $\partial N_Y \cap E = F(Y)$, which imply $N_X \in \mathcal{M}(X)$ and $N_Y \in \mathcal{M}(Y)$, respectively. The former implies $0 < w(M_X) - w(N_X) = w(P \cap M_X) - w(P \cap M_Y)$ whereas the latter implies $0 < w(M_Y) - w(N_Y) = w(P \cap M_Y) - w(P \cap M_X)$, a contradiction. ■

3.2 Representation with Circuit Families

Take an arbitrary matroid $\mathbf{M} = (E, \mathcal{I})$ and let $\mathcal{C}$ be its circuit family. A function $\delta : \mathcal{C} \to E$ is called a *de-cycle function* if it satisfies $\delta(C) \in C$ for each $C \in \mathcal{C}$.

Let $F : 2^E \to 2^E$ be a matroidal choice function on $\mathbf{M}$. For each $C \in \mathcal{C}$, we have $F(C) \in \mathcal{B}(C)$, and hence $|F(C)| = r(C) = |C| - 1$. Then, the set $C \setminus F(C)$ is a singleton. Define a de-cycle function $\delta_F : \mathcal{C} \to E$ by letting $\delta_F(C)$ be the only element in $C \setminus F(C)$ for each $C \in \mathcal{C}$. That is, identifying a singleton with its element, a de-cycle function δ_F is defined by

$$\delta_F(C) = C \setminus F(C) \quad (C \in \mathcal{C}).$$

We call δ_F the *associated* de-cycle function of F.

Lemma 3.8. Let $\delta_F : \mathcal{C} \to E$ be the associated de-cycle function of a matroidal choice function F on $\mathbf{M}$. Then, for any $X \subseteq E$ we have

$$F(X) = X \setminus \{\, \delta_F(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}.$$

Proof. Let $R(X) := \{\, \delta_F(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}$. For every $e \in R(X)$, there is $C \in \mathcal{C}$ such that $\delta_F(C) = e$ and $C \subseteq X$. Then $\{e\} = C \setminus F(C) \subseteq X \setminus F(X)$ by (Sub*). Therefore, $R(X) \subseteq X \setminus F(X)$ which implies $F(X) \subseteq X \setminus R(X)$. Also, the set $X \setminus R(X)$ is independent since $R(X)$ contains at least one element of each circuit. Thus $F(X) \subseteq X \setminus R(X) \in \mathcal{I}$. As $F(X) \in \mathcal{B}(X)$, this means $F(X) = X \setminus R(X)$, and the claim holds. □

Lemma 3.8 implies that, if a de-cycle function δ is associated with some matroidal choice function, then $X \setminus \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\} \in \mathcal{B}(X)$ for every $X \subseteq E$. For a general de-cycle function, however, this does not hold. It may eliminates too many elements as follows: Let δ be a de-cycle function defined on the circuit family of a uniform matroid of rank 1 on $E = \{e_1, e_2, e_3\}$. If δ indicates e_1, e_2, e_3 for circuits $\{e_1, e_2\}, \{e_2, e_3\}, \{e_1, e_3\}$, respectively, then $E \setminus \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq E \,\} = \emptyset$.

Definition 3.9. A de-cycle function $\delta : \mathcal{C} \to E$ is called *coherent* if

$$X \setminus \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\} \in \mathcal{B}(X)$$

holds for every $X \subseteq E$. ■

Theorem 3.10. A de-cycle function $\delta : \mathcal{C} \to E$ is coherent if and only if it is the associated de-cycle function of some matroidal choice function on $\mathbf{M}$. Also, such a matroidal choice function $F : 2^E \to 2^E$ is uniquely defined by

$$F(X) = X \setminus \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\} \quad (X \subseteq E). \tag{2}$$

Proof. The "if" part and the second claim follow from Lemma 3.8. For the "only if" part, assume that δ is coherent and define F by (2). Then, $X \setminus F(X) = \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}$ for each $X \subseteq E$ and then (Sub*) follows. Also the coherency implies $F(X) \in \mathcal{B}(X)$. Thus, F is a matroidal choice function, and we see that its associated de-cycle function is δ. □

This proposition says that there is a one-to-one correspondence between matroidal choice functions on $\mathbf{M}$ and coherent de-cycle functions on the circuit family of $\mathbf{M}$.

4 Algorithmic Characterization

In this section, we characterize the coherency of de-cycle functions via the behaviour of an online algorithm. We call an algorithm *online* if it receives its input piece by piece and makes decisions without future information. Let $\mathcal{C}$ be the circuit family of a matroid $\mathbf{M} = (E, \mathcal{I})$ and recall the following basic facts on circuits [19].

Fact 4.1. For an independent set I and an element $e \in E \setminus I$, the subset $I + e$ contains a unique circuit if $I + e \notin \mathcal{I}$. ■

We write $C(I|e)$ for such a unique circuit contained in $I + e$.

Fact 4.2. For any $X \subseteq E$, take $B \in \mathcal{B}(X)$ and $e \in E \setminus B$ arbitrarily. If $B + e \in \mathcal{I}$, then $B + e \in \mathcal{B}(X + e)$. Otherwise, $B + e - f \in \mathcal{B}(X + e)$ for every $f \in C(B|e)$. ■

For an arbitrary (not necessarily coherent) de-cycle function $\delta : \mathcal{C} \to E$, let us design an algorithm MCF(δ) as follows. An input of the algorithm is a sequence of (not necessarily distinct) elements of E with arbitrary length.

Algorithm: MCF(δ)
Input: $(e_1, e_2, \ldots, e_k)$

1. $J \leftarrow \emptyset$.
2. For $i = 1$ to k, do:
   (a) If $J + e_i \in \mathcal{I}$, then $J \leftarrow J + e_i$.
   (b) Otherwise, $J \leftarrow J + e_i - \delta(C(J|e_i))$.
3. Return J.

By Facts 4.1 and 4.2 we can observe that, at any point in the algorithm, J is a base of the set received until then. This implies the following lemma.

Lemma 4.3. For an input $(e_1, e_2, \ldots, e_k)$, the algorithm MCF(δ) returns a base of the set $\{e_1, e_2, \ldots, e_k\}$. ■

In general, the base returned by the algorithm differs depending on the order of the sequence. That is, the output for $(e_1, e_2, \ldots, e_k)$ and that for $(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{\tilde{k}})$ may differ even if $\{e_1, e_2, \ldots, e_k\} = \{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{\tilde{k}}\}$. If δ is coherent, however, the output is independent of the order as follows.

Lemma 4.4. For a sequence $(e_1, e_2, \ldots, e_k)$, let $X := \{e_1, e_2, \ldots, e_k\}$. If δ is coherent, the algorithm MCF(δ) returns the subset $F(X)$ defined by (2).

Proof. Let J^* be the output of the algorithm. By the algorithm, $e \in X \setminus J^*$ means that $\delta(C(J|e_i)) = e$ for some i. Since $C(J|e_i) \subseteq X$, the definition of $F(X)$ implies $e \in X \setminus F(X)$. Thus, $X \setminus J^* \subseteq X \setminus F(X)$, and so $J^* \supseteq F(X)$. This yields $J^* = F(X)$ since both J^* and $F(X)$ are bases of X. □

Recall the one-to-one correspondence between matroidal choice functions and coherent de-cycle functions in Theorem 3.10.

Corollary 4.5. Let F be a matroidal choice function on $\mathbf{M}$ and δ_F be its associated de-cycle function. For any $X \subseteq E$ and $(e_1, e_2, \ldots, e_k)$ with $\{e_1, e_2, \ldots, e_k\} = X$, the algorithm MCF($\delta_F$) returns $F(X)$. ■

Actually, the coherency is not only sufficient but also necessary condition for the independence of outputs from orders.

Theorem 4.6. A de-cycle function δ is coherent if and only if it satisfies the following condition: For any sequences $(e_1, e_2, \ldots, e_k)$ and $(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{\tilde{k}})$, MCF($\delta$) returns the same subset if $\{e_1, e_2, \ldots, e_k\} = \{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{\tilde{k}}\}$.

Proof. The "only if" part is shown in Lemma 4.4. We now show the "if" part. Take $X \subseteq E$ arbitrarily and let $J^* \subseteq X$ be the subset returned by MCF(δ) for every sequence $(e_1, e_2, \ldots, e_k)$ with $X := \{e_1, e_2, \ldots, e_k\}$. For each $C \in \mathcal{C}$ with $C \subseteq X$, consider a sequence $(e_1, e_2, \ldots, e_{|X|})$ such that $\{e_1, e_2, \ldots, e_{|C|}\} = C$ and $\{e_{|C|+1}, \ldots, e_{|X|}\} = X \setminus C$. Then, the output J^* can not contain $\delta(C)$ by the algorithm. Thus, $\{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\} \subseteq X \setminus J^*$, and hence $J^* \subseteq X \setminus \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}$. Note that $X \setminus \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}$ is independent since it has no circuit. Also $J^* \in \mathcal{B}(X)$ by Lemma 4.3. Thus $J^* = X \setminus \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\} \in \mathcal{B}(X)$. □

5 Local Characterization

In this section, we characterize the coherency of de-cycle functions by a local property. Let $\mathbf{M} = (E, \mathcal{I})$ be a matroid and $\mathcal{C}$ and r be its circuit family and rank function, respectively. We use the following facts in proofs.

Fact 5.1. A subset $X \subseteq E$ includes two or more circuits if and only if it satisfies $r(X) \le |X| - 2$. ■

Fact 5.2. For any $X \subseteq E$ and $e \in E \setminus X$, we have $r(X + e) = r(X) + 1$ if and only if there is no $C \in \mathcal{C}$ such that $e \in C \subseteq X + e$. ■

To describe our local characterization, we first introduce the concept of "minimal pair of circuits."

Definition 5.3. A pair of circuits $(C_1, C_2) \in \mathcal{C} \times \mathcal{C}$ is said to be *minimal* if it satisfies the following two conditions:

1. $C_1 \neq C_2$.
2. There exists no pair of circuits $(C'_1, C'_2) \in \mathcal{C} \times \mathcal{C}$ such that $C'_1 \neq C'_2$ and $(C'_1 \cup C'_2) \subsetneq (C_1 \cup C_2)$. ■

For a uniform matroid, a pair of circuits (C_1, C_2) is minimal if and only if $|C_1 \setminus C_2|$ $(= |C_2 \setminus C_1|) = 1$. For a graphic matroid, a pair of circuits (i.e., cycles) (C_1, C_2) is minimal if and only if it satisfies one of the following two conditions: (i) C_1 and C_2 are disjoint; (ii) $C_1 \cup C_2$ forms a theta graph, i.e., a graph which consists of three internally disjoint simple paths that have the same two distinct end vertices.

This minimality can also be represented by the rank function as follows.

Proposition 5.4. A pair of circuits (C_1, C_2) is minimal if and only if

$$r(C_1 \cup C_2) = |C_1 \cup C_2| - 2.$$

Proof. The "if" part: If (C_1, C_2) is minimal, then $C_1 \neq C_2$. Take $e_1 \in C_1 \setminus C_2$ and $e_2 \in C_2 \setminus C_1$ arbitrarily. Then, there is no circuit C such that $C \subseteq (C_1 \cup C_2) \setminus \{ e_1, e_2 \}$ since such C satisfies $C_1 \neq C$ and $(C_1 \cup C) \subsetneq (C_1 \cup C_2)$ which contradicts the minimality of (C_1, C_2). Hence $(C_1 \cup C_2) \setminus \{ e_1, e_2 \}$ is independent, so $r(C_1 \cup C_2) \geq |C_1 \cup C_2| - 2$. By Fact 5.1, the equality holds.

The "only if" part: Assume $r(C_1 \cup C_2) = |C_1 \cup C_2| - 2$. For any $e \in C_1 \cup C_2$, we have $r((C_1 \cup C_2) - e) = r(C_1 \cup C_2)$, and hence $r((C_1 \cup C_2) - e) = |(C_1 \cup C_2) - e| - 1$. Then, by Fact 5.1, $(C_1 \cup C_2) - e$ cannot include two distinct circuits. Therefore, (C_1, C_2) is minimal. □

We prepare the following lemma for our local characterization.

Lemma 5.5. For two distinct circuits C_1 and C_0 and arbitrary $e \in C_1$, there exists a circuit C_2 such that $e \notin C_2 \subseteq C_1 \cup C_0$ and (C_1, C_2) is minimal.

Proof. Let $C_2 \in \mathcal{C}$ such that $e \notin C_2 \subseteq C_1 \cup C_0$ and $|C_1 \cup C_2|$ is minimal. Then (C_1, C_2) is a minimal pair as follows. Suppose, to the contrary, it fails. Then $r(C_1 \cup C_2) \leq |C_1 \cup C_2| - 3$ by Fact 5.1 and Proposition 5.4. Take $e' \in C_2 \setminus C_1$ arbitrarily. Then $r(C_1 \cup C_2 \setminus \{e, e'\}) \leq r(C_1 \cup C_2) \leq |C_1 \cup C_2| - 3 = |C_1 \cup C_2 \setminus \{e, e'\}| - 1$. Hence, $C_1 \cup C_2 \setminus \{e, e'\}$ is dependent and so includes a circuit C_2', which satisfies $e \notin C_2' \subseteq C_1 \cup C_2 \subseteq C_1 \cup C_0$. Also $|C_1 \cup C_2'| \leq |C_1 \cup C_2 - e'| < |C_1 \cup C_2|$, a contradiction. □

Let $\delta : \mathcal{C} \to E$ be an arbitrary de-cycle function. Now we provide the main theorem of this section.

Theorem 5.6. δ is coherent if and only if δ satisfies the following condition:

(D) For a minimal pair $(C_1, C_2) \in \mathcal{C} \times \mathcal{C}$ and $C_3 \in \mathcal{C}$ with $C_3 \subseteq C_1 \cup C_2$,

$$|\{\delta(C_1), \delta(C_2), \delta(C_3)\}| \leq 2.$$

Proof. We respectively show the "only if" and "if" parts of the theorem in Lemmas 5.7 and 5.8 described below. □

If we regard the minimality of a pair of circuits as a kind of closeness, the condition (D) can be translated as follows: If three circuits are close to each other, then the same element is indicated for at least two of them.

Lemma 5.7. If δ is coherent, then δ satisfies (D).

Proof. Let δ be coherent and suppose, to the contrary, that $\delta(C_1)$, $\delta(C_2)$, and $\delta(C_3)$ are all distinct for a minimal pair (C_1, C_2) and a circuit $C_3 \subseteq C_1 \cup C_2$. Then $X := C_1 \cup C_2$ satisfies $|X \setminus \{\delta(C) : C \in \mathcal{C},\ C \subseteq X\}| \leq |X| - 3$ whereas Proposition 5.4 implies $r(X) = |X| - 2$. Thus, $X \setminus \{\,\delta(C) \mid C \in \mathcal{C},\ C \subseteq X\,\}$ can not be a base of X, and hence δ is not coherent. □

Lemma 5.8. If δ satisfies (D), then δ is coherent.

Proof. Assume that δ satisfies (D) and define $F(X)$ by (2) and let

$$R(X) := X \setminus F(X) = \{\,\delta(C) \mid C \in \mathcal{C},\ C \subseteq X\,\}$$

for every $X \subseteq E$. We show $F(X) \in \mathcal{B}(X)$ ($\forall X \subseteq E$), i.e., the coherency, using an induction. Clearly $F(\emptyset) = \emptyset \in \mathcal{B}(\emptyset)$. For any $X \subseteq E$ and $e \notin X$, let us show $F(X + e) \in \mathcal{B}(X + e)$ assuming $F(X) \in \mathcal{B}(X)$. By definition of F, the set $F(X + e)$ is independent, and hence what is left is to show $|F(X+e)| = r(X+e)$. We need consider two cases: (i) $r(X+e) = r(X)+1$, and (ii) $r(X + e) = r(X)$.

Case (i): By Fact 5.2, $r(X + e) = r(X) + 1$ implies that there is no $C \in \mathcal{C}$ with $e \in C \subseteq X + e$, and hence $R(X + e) = R(X)$. This implies $F(X + e) = F(X) + e$. Since $|F(X)| = r(X)$ by $F(X) \in \mathcal{B}(X)$, we have $|F(X + e)| = r(X) + 1 = r(X + e)$.

Case (ii): Since $r(X + e) = r(X) = |F(X)|$, we are reduced to show $|F(X+e)| = |F(X)|$, which is equivalent to $|R(X+e)| = |R(X)|+1$. Also, we have $R(X) \subseteq R(X + e)$ by definition. Then, it suffices to show that $R(X + e) \setminus R(X)$ is a singleton.

Since $F(X) \in \mathcal{B}(X)$ and $r(X) = r(X + e)$, the set $F(X) + e$ includes a circuit C_0 with $e \in C_0$. Since $C_0 \subseteq F(X) + e \subseteq X + e$, we have $\delta(C_0) \in R(X + e)$. Also, $C_0 \subseteq F(X) + e = X \setminus R(X) + e$ and $e \in E \setminus X$ implies

$$C_0 \cap R(X) = \emptyset, \tag{3}$$

and hence $\delta(C_0) \notin R(X)$. Therefore, $\delta(C_0) \in R(X + e) \setminus R(X)$. Next, we show that $R(X + e) \setminus R(X)$ contains only $\delta(C_0)$.

Suppose, to the contrary, that there is a circuit C_1 such that

$$C_1 \subseteq X + e, \quad \delta(C_1) \in R(X + e) \setminus R(X), \quad \delta(C_1) \neq \delta(C_0). \tag{4}$$

Note that $\delta(C_1) \notin R(X)$ implies $e \in C_1$ and $\delta(C_1) \neq \delta(C_0)$ implies $C_1 \neq C_0$. If multiple circuits satisfy these condition, let C_1 be the one which minimizes

$|C_1 \cap R(X)|$. By Lemma 5.5 for C_1, C_0 and $e \in C_1$, there exists $C_2 \in \mathcal{C}$ such that $e \notin C_2 \subseteq C_1 \cup C_0$ and (C_1, C_2) is minimal. Then $C_2 \subseteq C_1 \cup C_0 - e \subseteq X$, and hence $\delta(C_2) \in R(X)$, which implies $\delta(C_2) \notin C_0$ by (3). Then, $\delta(C_2) \in C_2 \subseteq C_1 \cup C_0$ implies $\delta(C_2) \in C_1$. Thus we obtain $\delta(C_2) \in C_1 \cap C_2$. By the axiom (C3) of circuits, there is a circuit C_3 such that

$$\delta(C_2) \notin C_3 \subseteq C_1 \cup C_2.$$

Since (C_1, C_2) is minimal and $C_3 \subseteq C_1 \cup C_2$, the condition (D) yields

$$|\{\delta(C_1), \delta(C_2), \delta(C_3)\}| \leq 2.$$

This implies $\delta(C_1) = \delta(C_3)$ since we have $\delta(C_1) \neq \delta(C_2)$ and $\delta(C_2) \neq \delta(C_3)$, which follow from $\delta(C_1) \notin R(X) \ni \delta(C_2)$ and $\delta(C_2) \notin C_3$, respectively. The fact $\delta(C_1) = \delta(C_3)$ implies that (4) holds with C_1 replaced by C_3. Also, $C_3 \subseteq C_1 \cup C_2 \subseteq C_1 \cup C_0$ and (3) implies $C_3 \cap R(X) \subseteq C_1 \cap R(X)$, whose equality fails by $\delta(C_2) \in (C_1 \cap R(X)) \setminus C_3$. Hence, $|C_3 \cap R(X)| < |C_1 \cap R(X)|$, which contradicts the minimality of $|C_1 \cap R(X)|$. □

6 Relationships with Discrete Concavity

This section shows that matroidal choice functions are closely related to valuated matroids and M$^\natural$-concave functions.

6.1 Valuated Matroids and M$^\natural$-concave Functions

A *valuated matroid*, introduced by Dress and Wenzel [2, 4], is a pair $(\mathcal{B}, \omega)$ such that $\mathcal{B}$ is the base family of a matroid and $\omega : \mathcal{B} \to \mathbf{R}$ is a function which satisfies the following exchange axiom.

(VM) For any $B_1, B_2 \in \mathcal{B}$ and $e_1 \in B_1 \setminus B_2$, there is $e_2 \in B_2 \setminus B_1$ such that $B_1 - e_1 + e_2,\ B_2 + e_1 - e_2 \in \mathcal{B}$ and

$$\omega(B_1) + \omega(B_2) \leq \omega(B_1 - e_1 + e_2) + \omega(B_2 + e_1 - e_2).$$

Murota and Shioura [16] generalized matroid valuations to M$^\natural$-concave functions on the integer lattice. In this paper, we only consider the M$^\natural$-concavity for set functions. For any $f : 2^E \to \mathbf{R} \cup \{-\infty\}$, define its effective domain by $\mathrm{dom}\, f := \{\, X \mid f(X) \neq -\infty \,\}$. We say that f is *M$^\natural$-concave* if is satisfies the following exchange axiom.

(M$^\natural$) For any $X_1, X_2 \in \mathrm{dom}\, f$, $e_1 \in X_1 \setminus X_2$, either of the following holds:

1. $f(X_1) + f(X_2) \leq f(X_1 - e_1) + f(X_2 + e_1)$,
2. $\exists e_2 \in X_2 \setminus X_1 : f(X_1) + f(X_2) \leq f(X_1 - e_1 + e_2) + f(X_2 + e_1 - e_2)$.

The following facts are known (see, e.g., [16]) and can be easily confirmed.

Fact 6.1. For an $\mathrm{M}^\natural$-concave function $f : 2^E \to \mathbf{R} \cup \{-\infty\}$ with $\emptyset \in \operatorname{dom} f$, the pair $(E, \operatorname{dom} f)$ is a matroid. ■

Fact 6.2. For an $\mathrm{M}^\natural$-concave function $f : 2^E \to \mathbf{R} \cup \{-\infty\}$ with $\emptyset \in \operatorname{dom} f$, let $\mathcal{B}$ be the base family of the matroid $(E, \operatorname{dom} f)$ and $f|_{\mathcal{B}} : \mathcal{B} \to \mathbf{R}$ be the restriction of f to $\mathcal{B}$. Then, $(\mathcal{B}, f|_{\mathcal{B}})$ is a valuated matroid. ■

The $\mathrm{M}^\natural$-concavity has the following local characterization.

Lemma 6.3 (Shioura and Tamura [21])**.** For $f : 2^E \to \mathbf{R} \cup \{-\infty\}$, assume that $(E, \operatorname{dom} f)$ is a matroid. Then, f is $\mathrm{M}^\natural$-concave if and only if f satisfies the following conditions for every $X \subseteq E$ and $a, b, c \in E \setminus X$:

1. $f(X) + f(X \cup \{a, b\}) \leq f(X + a) + f(X + b)$,
2. $f(X + a) + f(X \cup \{b, c\}) \leq$
   $\max\{f(X + b) + f(X \cup \{a, c\}), f(X + c) + f(X \cup \{a, b\})\}$,

where we admit the inequality of the form $-\infty \leq -\infty$. ■

6.2 Induced Matroidal Choice Functions

Let $f : 2^E \to \mathbf{R} \cup \{-\infty\}$ be a value function. That is, we regard $X \in \operatorname{dom} f$ as an acceptable set with value $f(X)$ and $X \in 2^E \setminus \operatorname{dom} f$ as unacceptable. We assume $\emptyset \in \operatorname{dom} f$. A value function f is called *unique-selecting* if for each $X \subseteq E$ the maximum value of f among all subsets of X is attained by a unique subset, which we denote by $\arg\max\{\, f(Y) \mid Y \subseteq X \,\}$. For a unique-selecting function f, the choice function $F : 2^E \to 2^E$ defined by

$$F(X) = \arg\max\{\, f(Y) \mid Y \subseteq X \,\} \quad (X \subseteq E)$$

is said to be *induced from* f. The following fact is implied by Lemma 5.2 of Fujishige and Tamura [8] (see also Lemma 3.3 of [18]).

Lemma 6.4. Let f be a unique-selecting $\mathrm{M}^\natural$-concave function. Then, the choice function F induced from f is substitutable. ■

We call f *monotone* if $X \subseteq Y$ implies $f(X) \leq f(Y)$ for $X, Y \in \operatorname{dom} f$.

Theorem 6.5. Let F be a choice function induced from a unique-selecting $\mathrm{M}^\natural$-concave function f. If f is monotone, then F is a matroidal choice function on the matroid $(E, \operatorname{dom} f)$.

Proof. The substitutability follows from Lemma 6.4. The pair $(E, \operatorname{dom} f)$ is a matroid by Fact 6.1. For any $X \subseteq E$, the monotonicity of f implies $F(X) \in \mathcal{B}(X)$ where $\mathcal{B}(X)$ denotes the family of bases of X. □

This Theorem 6.5 will be generalized in Theorem 6.7 using the following Lemma 6.6. For a choice function $F : 2^E \to 2^E$, we denote by $\mathcal{A}(F)$ the family of accepted subsets, i.e., $\mathcal{A}(F) := \{ X \subseteq E \mid F(X) = X \}$.

Lemma 6.6. Let F be a choice function induced from a unique-selecting M$^\natural$-concave function f. If $(E, \mathcal{A}(F))$ is a matroid, then a function $\hat{f} : 2^E \to \mathbf{R} \cup \{-\infty\}$ defined by

$$\hat{f}(X) = \begin{cases} f(X) & X \in \mathcal{A}(F), \\ -\infty & \text{otherwise} \end{cases} \tag{5}$$

is a monotone M$^\natural$-concave function with $\operatorname{dom} \hat{f} = \mathcal{A}(F)$.

Proof. We have $\operatorname{dom} \hat{f} = \mathcal{A}(F)$ by definition. Then any $X \in \operatorname{dom} \hat{f}$ satisfies $F(X) = X$, which implies $f(Y) < f(X)$ $(\forall Y \subsetneq X)$, and hence $\hat{f}$ is monotone. What is left is to show the M$^\natural$-concavity of $\hat{f}$. It suffices to show that Conditions 1 and 2 in Lemma 6.3 hold with f replaced by $\hat{f}$. Take any $X \subseteq E$ and $a, b, c \subseteq E \setminus X$. We fist show Condition 1, i.e.,

$$\hat{f}(X) + \hat{f}(X \cup \{a, b\}) \leq \hat{f}(X + a) + \hat{f}(X + b). \tag{6}$$

We assume $X \cup \{a, b\} \in \mathcal{A}(F)$, since otherwise $\hat{f}(X \cup \{a, b\}) = -\infty$ and (6) holds obviously. Then, $\{X,\ X + a,\ X + b,\ X \cup \{a, b\}\} \subseteq \mathcal{A}(F)$ follows by the axiom (I2) of the matroid $(E, \mathcal{A}(F))$. Then, $\hat{f}$ in the inequality (6) can be replaced by f, and then the inequality holds since f is M$^\natural$-concave.

We next show Condition 2, i.e.,

$$\begin{aligned} &\hat{f}(X + a) + \hat{f}(X \cup \{b, c\}) \leq \\ &\qquad \max\{\hat{f}(X + b) + \hat{f}(X \cup \{a, c\}), \hat{f}(X + c) + \hat{f}(X \cup \{a, b\})\}. \end{aligned} \tag{7}$$

We assume that a, b, c are all distinct and $X + a, X \cup \{b, c\} \in \mathcal{A}(F)$, since otherwise (7) is obvious. Then, $X + b, X + c \in \mathcal{A}(F)$ by the axiom (I2) of the matroid $(E, \mathcal{A}(F))$. Also, by applying (I3) to $X + a, X \cup \{b, c\} \in \mathcal{A}(F)$, we obtain

$$X \cup \{a, b\} \in \mathcal{A}(F) \quad \text{or} \quad X \cup \{a, c\} \in \mathcal{A}(F). \tag{8}$$

Note that the inequality (7) with $\hat{f}$ replaced by f holds. Without loss of generality, we assume $f(X + b) + f(X \cup \{a, c\})$ is larger than or equal to $f(X + c) + f(X \cup \{a, b\})$. Then, we have

$$f(X + c) + f(X \cup \{a, b\}) \leq f(X + b) + f(X \cup \{a, c\}), \tag{9}$$

$$f(X + a) + f(X \cup \{b, c\}) \leq f(X + b) + f(X \cup \{a, c\}). \tag{10}$$

To obtain a contradiction, suppose that (7) fails for $\hat{f}$. Then, we have

$$\hat{f}(X + a) + \hat{f}(X \cup \{b, c\}) > \hat{f}(X + b) + \hat{f}(X \cup \{a, c\}). \tag{11}$$

Since we have $\hat{f}(Y) = f(Y)$ for each $Y \in \{X+a, X+b, X \cup \{b,c\}\} \subseteq \mathcal{A}(F)$, (10) and (11) imply $f(X \cup \{a,c\}) \neq \hat{f}(X \cup \{a,c\})$. Hence $X \cup \{a,c\} \notin \mathcal{A}(F)$. With (8), this yields $X \cup \{a,b\} \in \mathcal{A}(F)$. Then, $X \cup \{b,c\}, X \cup \{a,b\} \in \mathcal{A}(F)$ and $X \cup \{a,c\} \notin \mathcal{A}(F)$ respectively imply

$$f(X \cup \{b,c\}) > f(X+b), \tag{12}$$

$$f(X \cup \{a,b\}) > f(X+b), \tag{13}$$

$$Y^* := \arg\max\{\, f(Y) \mid Y \subseteq X \cup \{a,c\} \,\} \neq X \cup \{a,c\}. \tag{14}$$

There are three cases (i) $c \notin Y^*$, (ii) $a \notin Y^*$, (iii) $\{a,c\} \subseteq Y^*$, $X \setminus Y^* \neq \emptyset$.

Case (i): $c \notin Y^* \subseteq X \cup \{a,c\}$ implies $Y^* \subseteq X+a$. Since $X+a \in \mathcal{A}(F)$, means $F(X+a) = X+a$, this implies $Y^* = X+a$. Then, (14) implies $f(X+a) > f(X \cup \{a,c\})$. Combined with (12), this contradicts (10).

Case (ii): By a similar argument, $a \notin Y^* \subseteq X \cup \{a,c\}$ implies $f(X+c) > f(X \cup \{a,c\})$. Combined with (13), this contradicts (9).

Case (iii): We have $\{a,c\} \subseteq Y^* \subseteq X \cup \{a,c\}$. Apply the exchange axiom ($\mathrm{M}^\natural$) of f to Y^* and $X+a$ and $c \in Y^* \setminus (X+a)$. Then, we have

$$f(Y^*) + f(X+a) \leq f(Y^* - c) + f(X \cup \{a,c\}) \tag{15}$$

or there is $d \in (X+a) \setminus Y^* = X \setminus Y^*$ such that

$$f(Y^*) + f(X+a) \leq f(Y^* - c + d) + f((X \cup \{a,c\}) - d). \tag{16}$$

By $Y^* - c \subseteq X+a$ and $X+a \in \mathcal{A}(F)$, we have $f(X+a) > f(Y^* - c)$. In the case where (15) holds, this implies $f(Y^*) < f(X \cup \{a,c\})$, which contradicts (14). Similarly, in the case where (16) holds, since $Y^* - c + d \subseteq X+a$ and $X+a \in \mathcal{A}(F)$ imply $f(X+a) > f(Y^* - c + d)$, we obtain $f(Y^*) < f((X \cup \{a,c\}) - d)$, which also contradicts (14). □

Theorem 6.7. Let F be a choice function induced from a unique-selecting $\mathrm{M}^\natural$-concave function f. If $(E, \mathcal{A}(F))$ is a matroid, then F is a matroidal choice function on $(E, \mathcal{A}(F))$.

Proof. By Lemma 6.6, $\hat{f}$ defined by (5) is a monotone $\mathrm{M}^\natural$-concave function with $\operatorname{dom} \hat{f} = \mathcal{A}(F)$. Observe that $\hat{f}$ also induces F. Then, Theorem 6.5 implies that F is a matroidal choice function on $(E, \mathcal{A}(F))$. □

Lemma 6.6, combined with Fact 6.2, also implies the following corollary, which is used in the following section.

Corollary 6.8. Let F be a choice function induced from a unique-selecting $\mathrm{M}^\natural$-concave function f. If $(E, \mathcal{A}(F))$ is a matroid, then $(\mathcal{B}, f|_{\mathcal{B}})$ is a valuated matroid, where $\mathcal{B}$ is the base family of $(E, \mathcal{A}(F))$.

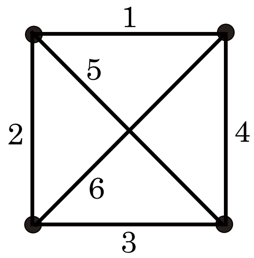

Figure 1: A graphical representation of $\mathbf{M}$

6.3 Uninducible Matroidal Choice Function

As shown in Theorem 6.7, an M$^\natural$-concave function induces a matroidal choice function under certain conditions. Then, one may wonder if every matroidal choice function is obtained in such a way. This is not true in general. We give a counter example using the following proposition.

Proposition 6.9. (Dress and Wenzel [4]) Let $(\mathcal{B}, \omega)$ be a valuated matroid on $\mathbf{M}$. If $\mathbf{M}$ is a binary matroid, then ω can be represented as

$$\omega(B) = \alpha + \sum_{e \in B} \eta(e) \quad (B \in \mathcal{B})$$

for some $\alpha \in \mathbf{R}$ and $\eta : E \to \mathbf{R}$. ■

Example 6.10. We give an example of matroidal choice function which cannot be induced from any M$^\natural$-concave set function in the way described in Section 6.2. Let $\mathbf{M} = (E, \mathcal{I})$ be a matroid such that $E = \{e_1, e_2, \ldots, e_6\}$ and $\mathcal{C}$ consists of the following seven circuits:

$$C_1 = \{e_1, e_2, e_3, e_4\}, \quad C_2 = \{e_1, e_3, e_5, e_6\}, \quad C_3 = \{e_2, e_4, e_5, e_6\},$$
$$C_4 = \{e_1, e_2, e_6\}, \quad C_5 = \{e_2, e_3, e_5\}, \quad C_6 = \{e_3, e_4, e_6\}, \quad C_7 = \{e_1, e_4, e_5\}.$$

This is a graphic matroid. (Figure 1 shows its graphical representation.)

Define a de-cycle function $\delta : \mathcal{C} \to E$ as

$$\begin{aligned}
&\delta(C_1) = e_4, \\
&\delta(C_2) = \delta(C_4) = \delta(C_6) = e_6, \\
&\delta(C_3) = \delta(C_5) = \delta(C_7) = e_5.
\end{aligned}$$

We can check that d is coherent. Hence, F defined by (2) is a matroidal choice function on $\mathbf{M}$ and $\mathcal{A}(F) = \mathcal{I}$ by Proposition 3.2. We now show that there is no unique-selecting M$^\natural$-concave function $f : 2^E \to \mathbf{R} \cup \{-\infty\}$ which satisfies

$$\forall X \subseteq E : \ F(X) = \arg\max \{ f(Y) \mid Y \subseteq X \}.$$

Suppose, to the contrary, there exists such a function f. Since f is M$^\natural$-concave and $(E, \mathcal{A}(F)) = \mathbf{M}$ is a matroid, Corollary 6.8 implies that $(\mathcal{B}, f|_{\mathcal{B}})$ is a valuated matroid, where $\mathcal{B}$ is the base family of $\mathbf{M}$. Since $\mathbf{M}$ is graphic, and hence binary, Proposition 6.9 implies that

$$\forall B \in \mathcal{B}: \ f(B) = \alpha + \textstyle\sum_{e \in B} \eta(e)$$

holds for some $\alpha \in \mathbf{R}$ and $\eta : E \to \mathbf{R}$. Note that, for $C_2 = \{e_1, e_3, e_5, e_6\}$, we have $F(C_2) = C_2 - \delta(C_2) = \{e_1, e_3, e_5\}$, which implies $f(\{e_1, e_3, e_5\}) > f(\{e_1, e_3, e_6\})$. As $\{e_1, e_3, e_5\}, \{e_1, e_3, e_6\} \in \mathcal{B}$, this implies $\eta(e_5) > \eta(e_6)$. On the other hand, $C_3 = \{e_2, e_4, e_5, e_6\}$ and $F(C_3) = C_3 - \delta(C_3) = \{e_2, e_4, e_6\}$ imply $f(\{e_2, e_4, e_6\}) > f(\{e_2, e_4, e_5\})$, and hence $\eta(e_6) > \eta(e_5)$, a contradiction. ■

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