A Characterization of Locally Testable Affine-Invariant Properties via Decomposition Theorems

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Property Testing

Definition

$f : \{0,1\}^n \to \{0,1\}$ is *ϵ-far* from $\mathcal{P}$ if, for any $g : \{0,1\}^n \to \{0,1\}$ satisfying $\mathcal{P}$,

$$\Pr_x[f(x) \neq g(x)] \geq \epsilon.$$

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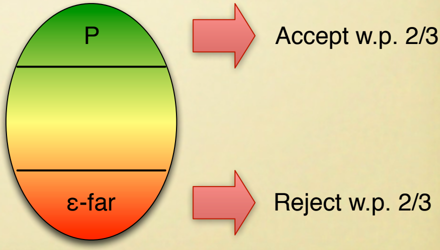
ϵ-tester for a property $\mathcal{P}$:

- Given $f : \{0,1\}^n \to \{0,1\}$ as a query access.
- Proximity parameter $\epsilon > 0$.



Local Testability and Affine-Invariance

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$\mathcal{P}$ is *affine-invariant* if a function $f : \mathbb{F}_2^n \to \{0,1\}$ satisfies $\mathcal{P}$, then $f \circ A$ satisfies $\mathcal{P}$ for any bijective affine transformation $A : \mathbb{F}_2^n \to \mathbb{F}_2^n$.

Examples of locally testable affine-invariant properties:

- d-degree Polynomials [AKK$^+$05, BKS$^+$10].
- Fourier sparsity [GOS$^+$11].
- Odd-cycle-freeness [BGRS12].

The Question and Related Work

Q. Characterization of locally testable affine-invariant properties? [KS08]

- Locally testable with one-sided error $\Leftrightarrow$ affine-subspace hereditary? [BGS10]
  Ex. low-degree polynomials, odd-cycle-freeness.
  - $\Rightarrow$ is true. [BGS10]
  - $\Leftarrow$ is true (if the property has bounded complexity). [BFH+13].

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- $\mathcal{P}$ is locally testable $\Rightarrow$ distance to $\mathcal{P}$ is estimable. [HL13]
- $\mathcal{P}$ is locally testable $\Leftrightarrow$ regular-reducible. [This work]

Graph Property Testing

Definition

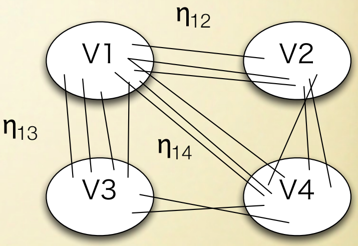
A graph $G = (V, E)$ is *ϵ-far* from a property $\mathcal{P}$ if we must add or remove at least $\epsilon|V|^2$ edges to make G satisfy $\mathcal{P}$.

Examples of locally testable properties:

- 3-Colorability [GGR98]
- H-freeness [AFKS00]
- Monotone properties [AS08b]
- Hereditary properties [AS08a]

A Characterization of Locally Testable Graph Properties

Szemerédi's regularity lemma:
Every graph can be partitioned into a constant number of parts so that each pair of parts looks random.



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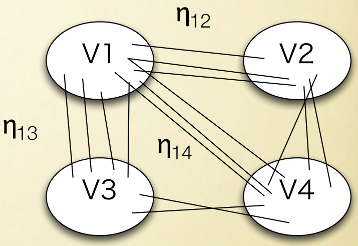

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A graph property $\mathcal{P}$ is locally testable
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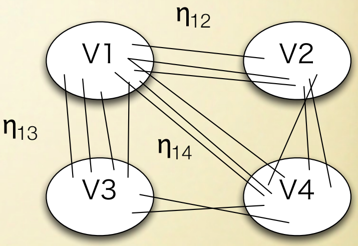

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Q. How can we extract such constant-size sketches from functions?

Constant Sketch for Functions

Theorem (Decomposition Theorem [BFH+13])

For any $\gamma > 0$, $d \geq 1$, and $r : \mathbb{N} \to \mathbb{N}$, there exists $\overline{C}$ such that: any function $f : \mathbb{F}_2^n \to \{0, 1\}$ can be decomposed as $f = f' + f''$, where

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- *a structured part $f' : \mathbb{F}_2^n \to [0,1]$, where*
  - *$f' = \Gamma(P_1, \ldots, P_C)$ with $C \leq \overline{C}$,*
  - *$P_1, \ldots, P_C$ are "non-classical" polynomials of degree $< d$ and rank $\geq r(C)$.*
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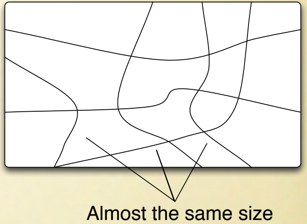
  - *$\Gamma : \mathbb{T}^C \to [0,1]$ is a function.*
- *a* **pseudo-random part** *$f'' : \mathbb{F}_2^n \to [-1,1]$*
  - *The Gowers norm $\|f''\|_{U^d}$ is at most γ.*

Factors

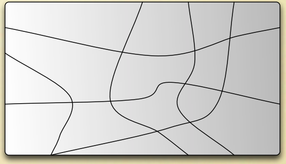
Polynomial sequence $(P_1, \ldots, P_C)$ partitions $\mathbb{F}_2^n$ into atoms $\{x \mid P_1(x) = b_1, \ldots, P_C(x) = b_C\}$.

$\mathbb{F}_2^n =$



The decomposition theorem says:

$f =$



$+\Upsilon$

$\Gamma(P_1, \ldots, P_C)$

What is the Gowers Norm?

Definition

Let $f : \mathbb{F}_2^n \to \mathbb{C}$. The *d-th Gowers norm* of f is

$$\|f\|_{U^d} := \Big(\mathop{\mathbf{E}}_{x, y_1, \ldots, y_d} \prod_{I \subseteq \{1,\ldots,d\}} J^{|I|} f(x + \sum_{i \in I} y_i) \Big)^{1/2^d},$$

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where J denotes complex conjugation.

- For any linear function $L : \mathbb{F}_2^n \to \mathbb{F}_2$,

$$\|(-1)^L\|_{U^2} = |\mathop{\mathbf{E}}_{x,y_1,y_2} (-1)^{L(x)+L(x+y_1)+L(x+y_2)+L(x+y_1+y_2)}| = 1.$$

- For any polynomial $P : \mathbb{F}_2^n \to \mathbb{F}_2$ of degree $< d$, $\|(-1)^P\|_{U^d} = 1$.

Gowers Norm Measures Correlation with Non-Classical Polynomials

Definition

$P : \mathbb{F}_2^n \to \mathbb{T}$ is a *non-classical polynomial of degree* $< d$ if $\|e^{2\pi i \cdot P}\|_{U^d} = 1$.

The range of P is $\{0, \frac{1}{2^{k+1}}, \ldots, \frac{2^{k+1}-1}{2^{k+1}}\}$ for some k (= *depth*).

Lemma

$f : \mathbb{F}_2^n \to \mathbb{C}$ with $\|f\|_\infty \leq 1$.

- $\|f\|_{U^d} \leq \epsilon \Rightarrow \langle f, e^{2\pi i \cdot P} \rangle \leq \epsilon$ *for any non-classical polynomial P of degree $< d$.* **(Direct Theorem)**
- $\|f\|_{U^d} \geq \epsilon \Rightarrow \langle f, e^{2\pi i \cdot P} \rangle \geq \delta(\epsilon)$ *for some non-classical polynomial P of degree $< d$.* **(Inverse Theorem)**

Is This Really a Constant-size Sketch?

- Structured part: $f' = \Gamma(P_1, \ldots, P_C)$.
- Γ indeed has a constant-size representation, but $P_1, \ldots, P_C$ may not have (even if we just want to specify the coset $\{P \circ A\}$).
- The rank of $(P_1, \ldots, P_C)$ is high
  $\Rightarrow$ Their degrees and depths determine almost everything.
  Ex. the distribution of the restriction of f to a random affine subspace.

Regularity-Instance

Formalize "f has some specific structured part".

Definition

A *regularity-instance* I is a tuple of

- an error parameter $\gamma > 0$,
- a structure function $\Gamma : \mathbb{T}^C \to [0, 1]$,
- a complexity parameter $C \in \mathbb{N}$,
- a degree-bound parameter $d \in \mathbb{N}$,
- a degree parameter $\mathbf{d} = (d_1, \ldots, d_C) \in \mathbb{N}^C$ with $d_i < d$,
- a depth parameter $\mathbf{h} = (h_1, \ldots, h_C) \in \mathbb{N}^C$ with $h_i < \frac{d_i}{p-1}$, and
- a rank parameter $r \in \mathbb{N}$.

Satisfying a Regularity-Instance

Definition

Let $I = (\gamma, \Gamma, C, d, \mathbf{d}, \mathbf{h}, r)$ be a regularity-instance.
f *satisfies* I if it is of the form

$$f(x) = \Gamma(P_1(x), \ldots, P_C(x)) + \Upsilon(x),$$

where

- P_i is a polynomial of degree d_i and depth h_i,
- $(P_1, \ldots, P_C)$ has rank at least r,
- $\|\Upsilon\|_{U^d} \le \gamma$.

Testing the Property of Satisfying a Regularity-Instance

Theorem

The property of satisfying a regularity-instance is locally testable (if the rank parameter is sufficiently large).

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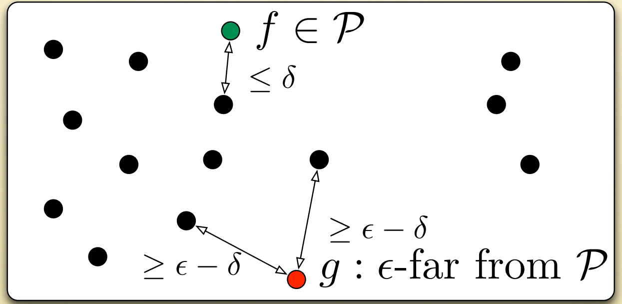
Algorithm:

1: Set δ small enough and m large enough.
2: Take a random affine embedding $A : \mathbb{F}_2^m \to \mathbb{F}_2^n$.
3: **if** $f \circ A$ is δ-close to satisfying I **then** accept.
4: **else** reject.

Regular-Reducibility

A property $\mathcal{P}$ is *regular-reducible* if for any $\delta > 0$, there exists a set $\mathcal{R}$ of constant number of high-rank regularity-instances with constant parameters such that:



Our Characterization

Theorem

An affine-invariant property $\mathcal{P}$ is locally testable

$$\Updownarrow$$

$\mathcal{P}$ is regular-reducible.

Proof Intuition

- Regular-reducible ⇒ Locally testable
  Combining the testability of regularity-instances and [HL13], we can estimate the distance to $\mathcal{R}$.
- Locally testable ⇒ Regular-reducible
  The behavior of a tester depends only on the distribution of the restriction to a random affine subspace. Since Γ, $\mathbf{d}$, and $\mathbf{h}$ determines the distribution, we can find $\mathcal{R}$ using the tester.

Conclusions

Obtained a characterization of locally testable affine-invariant properties.

- Easily extendable to $\mathbb{F}_p$ for a prime p.
- Query complexity is actually unknown due to the Gowers inverse theorem. Other parts involve Ackermann-like functions.

Open Problems

- Characterization based on function (ultra)limits?
- Characterization of linear-invariant properties?
- Study other groups?
  - Abelian $\Rightarrow$ Higher order Fourier analysis developed by [Sze12].
  - Non-Abelian $\Rightarrow$ Representation theory.
- Why is affine invariance easier to deal with than permutation invariance?