**Nonlinear Laplacian for Digraphs and its Applications to Network Analysis**

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**Question: Can we develop spectral graph theory for digraphs?**

- Spectral graph theory analyzes graph properties via eigenpairs of associated matrices (in particular, Laplacian).
- Well established for undirected graphs.
- Extensions for digraphs are largely unexplored although many real-world networks are directed.

### Definition

**The Laplacian for an undirected graph G**

- Adjacency matrix $A_G$
- Degree Matrix $D_G$
- Laplacian $L_G := D_G - A_G$
- Normalized Laplacian: $\mathcal{L}_G := D_G^{-1/2}L_G D_G^{-1/2}$

![Graph example](image)

**The Laplacian for a digraph G (proposed)**

- Laplacian $L_G : \mathbb{R}^V \rightarrow \mathbb{R}^V$ transforms $x \in \mathbb{R}^V$ as follows:
  - Construct an undirected graph $H$ as follows:
  - For each arc $u \rightarrow v$:
    - If $x(u) \geq x(v)$, add an undirected edge $[u, v]$.
    - Otherwise, add self-loops to $u$ and $v$.
  - Output $L_Hx$.

- Normalized Laplacian: $\mathcal{L}_G : x \mapsto D_G^{-1/2}L_G D_G^{-1/2}x$

### Interpretation via electrical circuits

- Regard $G$ as an electrical circuit.
  - Graph: edge = resistance of 1Ω
  - Digraph: arc = diode of 1Ω (current flows only one way)

For each $u \in V$, flow a current of $b(u)$ amperes to $u$.

The voltages $x \in \mathbb{R}^V$ of vertices is given by $L_G(x) = b$.

### Properties

- $(\lambda, v)$ is an eigenpair of $L_G$ if $L_G(v) = \lambda v$
- Trivial eigenpair $(\lambda_1, v_1)$ with $\lambda_1 = 0$.

For any subspace $U$ of positive dimension, $\Pi_U L_G$ has an eigenpair, $(\Pi_U = $ Projection matrix to $U$)

- Another eigenvalue of $L_G$ exists by choosing $U = v_1^\perp$.
  - Let $\lambda_2$ be the second smallest eigenvalue.

### Algorithm

- Computing $\lambda_2$ is (likely to be) NP-hard.
- Suppose we start the diffusion process
  
  $$dx = -\Pi_U L_G(x) dt$$

  from a vector in the subspace $U = v_1^\perp$.
  
  - $x$ converges to an eigenvector orthogonal to $v_1$.
  - Rayleigh quotient never increases during the process.

### Visualization

- Friendship network at a high school in Illinois
  - $(u \rightarrow v \mid u$ regards $v$ as a friend)$
  - Reorder vertices according to the eigenvector computed by the diffusion process

![Chung's Laplacian](image)

**Proposed Laplacian**

- $\lambda_2$ is the minimum of
  
  $$\sum_{u \rightarrow v} (x(u) - x(v))^2 \frac{\pi_u}{d_u^+}$$

  subject to $x \neq 0, x \perp v_1$.

### Community Detection

- (Directed) conductance $\phi^+(S)$ of $S$:
  
  $$\frac{\min(\text{vol}(S), \text{vol}(V-S))}{\text{vol}(S) \cdot \text{total degree of } u \in S}$$

- $\text{cut}^+(S)$: # of arcs from $S$ to $V-S$

- Cheeger's inequality for digraphs:
  
  $$\lambda_2 \leq \min_S \phi^+(S) \leq 2\sqrt{\lambda_2}$$

- Conductance of the set of the first $k$ vertices after reordering.

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