Making smallest-diameter graphs at “Graph Golf”

Takayuki Matsuzaki
Graduate School of Science and Technology, Kumamoto University

Teruaki Kitasuka
Graduate School of Engineering, Hiroshima University

Masahiro Iida
Faculty of Advanced Science and Technology, Kumamoto University

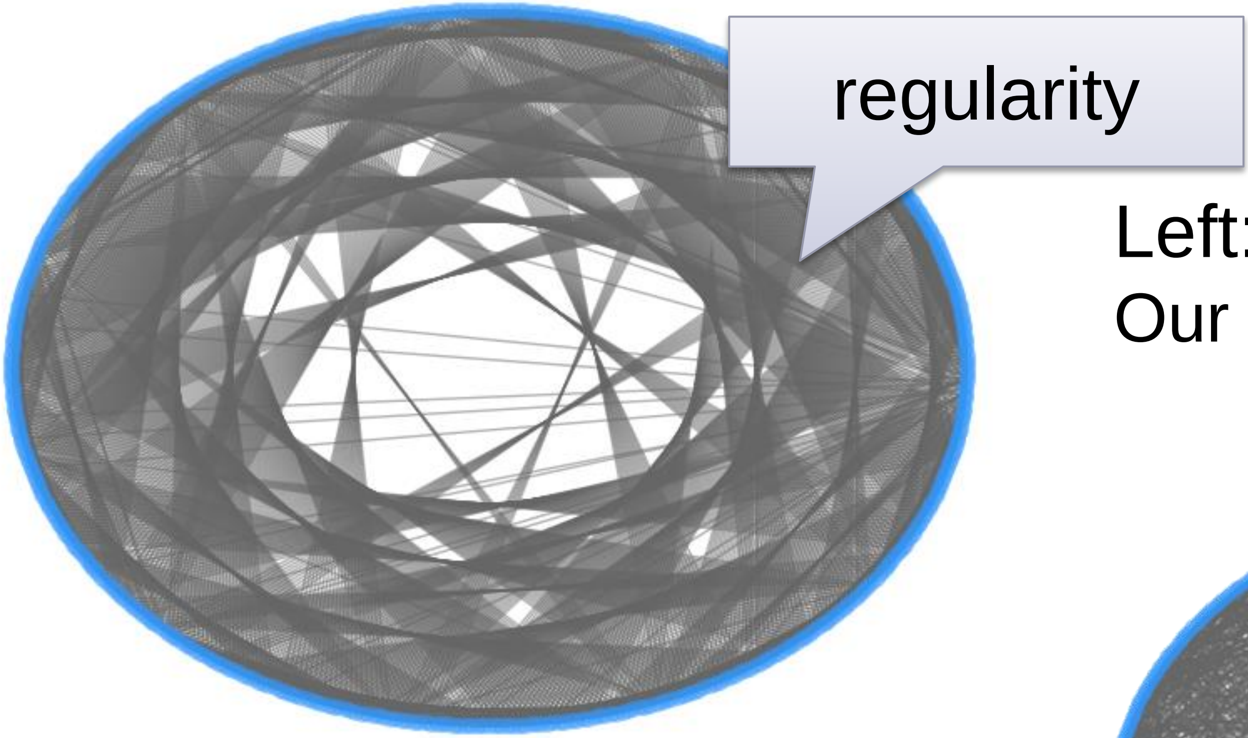
Nov. 22, 2016. CANDAR'16. Hiroshima, Japan.

Background

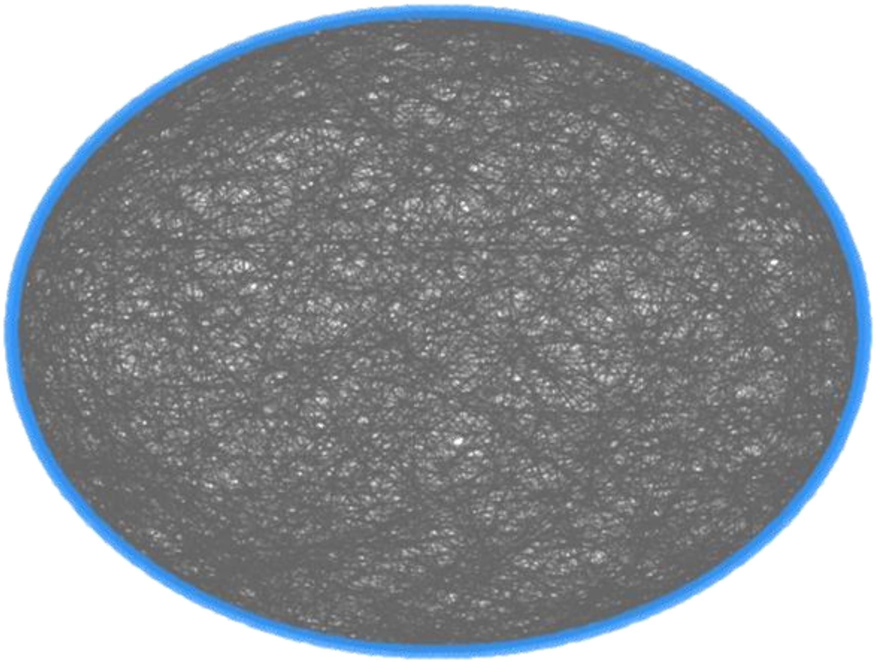
- "Graph Golf"
  - Aims at finding smallest-diameter graphs
  - We submitted eleven graphs, and won widest improvement award by using our approach

- Order/Degree problem
  - Given order/degree
  - Find a graph with minimum diameter and average shortest path length (ASPL)

Submissions: $n = 512, d = 8$



Left: $k = 4$, $l = 3.11138$
Our graph



Right: $k = 5$, $l = 3.26793$
Random graph

Figures are from GraphGolf 2016 website. (http://research.nii.ac.jp/graphgolf/ranking.html)

Why does the difference occur?

- We use Cayley graphs as a base
  - Studied in Degree/Diameter problem
  - Derived by mathematical way
  - It has regularity but not fit in ODP

We will explain Cayley graphs later.

Degree/diameter problem (DDP)

Given degree/diameter and find a graph with largest order

- Has been studied for a long time
- Many solutions are found
- Not always be applied for ODP solutions

Motivation

- Derive ODP from DDP
  - It will be useful if some alternation were applied
  - By adding (or removing) nodes/edges
  - Turn into desired order/degree graph which has smallest-diameter
- Two DDP outcomes
  - Brown graphs, Cayley graphs

How to make small graphs

To get small graphs among various order/degree pairs, we use following two methods.

1. Graph uniting method

   Using Brown graphs

   Applied to: (1 024, 32), (1 560, 40), (3 250, 57)

2. Node adding method

   Using Cayley graphs

   Applied to: (256, 8), (512, 8), (1 024, 8)
   (10 000, 7), (10 000, 11), (10 000, 20), (100 000, 20)

   Note: Base graph of (256, 8) is degree/diameter solution (n = 253, d = 8). So, node adding only.
   (1 024, 8)-graph is gained directly by Cayley graph.

Why we interested in DDP

Heuristics for diameter 3 graphs

- Create a base graph
  - Targeted diameter $k = 3$
  - Multiple Petersen graphs are connected
- Greedily add edges
  - To increase the # of pentagons

We described detail in GraphGolf 2015

Before the competition began

Using heuristics and 2-opt search …

- We made (36, 3)-graph
  - Submitted it in June 24.
  - Tied for first place

- Other graphs
  - Complete defeat

Its problems

- We can only make graphs $k = 3$

  In GraphGolf 2016, graphs of
  (64, 8), (300, 7), (1 024, 32), (1 560, 40), (3 250, 57) has $k = 3$

- Remainder 15 graphs
  - $k = 4$: (256, 8), (300, 7), (512, 8), (1 024, 11) , (10 000, 20)
  - $k = 5$: (1 024, 8), (1 800, 7), (10 000, 11), (100 000, 20)
  - $k = 6$: (96,3), (100 000, 11)
  - $k = 7$: (10 000, 7)
  - $k = 8$: (100 000, 7)
  - $k = 9$: (384, 3)
  - $k = 11$: (1024, 3)

Widest Improvement is difficult to attain.

What can we do in the competition?

Also, we focused on Deepest Improvement

However

- We don’t have enough computation power
- Not professional in graph theory

No idea to get another graphs

We depended on knowledge from DDP outcomes.

Degree/Diameter problem

DDP outcomes

We looked at

- Degree/Diameter problem
  - (253, 8)-graph
  - Its $k = 3$, $l = 2.730$

Can be expected

While in GraphGolf

- Random (256, 8)-graph
- Its $k = 4$, $l = 2.88$

Idea

- The difference is order
  - (253, 8)-graph has good k and l.
- Change order
  - By adding nodes/edges randomly
- Iterate it
  - Make many graphs
  - Leave the smallest-diameter one

Result of (256,8)-graph

- Then we got
  - $k = 4, l = 2.743$
  - Faster than making it from random

about 10 minutes

- Finally
  - Use 2-opt to it
  - $k = 4, l = 2.735$

few weeks

256 nodes, degree 8

Rank	Author	Diam.	ASPL
1	Our team	4	2.73597
2	Ryuhei Mori	4	2.74859

21 days

Experience from (256,8)-graph

- DDP solution can be used
  - As a good base graph
  - Studied for a long time
  - Many solutions are found

It seems to work out well.
We set goal to Widest Improvement.

Method 1

Graph uniting method

Attempt to make another graph

- Inspired by Brown’s construction

  Described in [1] at GraphGolf 2015

- It can make a graph $B(p)$

  Order = $p^{2k} + p^k + 1$

  Degree = $p^k + 1$

  Diameter = 2

  p: a prime   k: a natural number

[1] R. Mizuno and Y Ishida, "The construction of a regular graph," http://research.nii.ac.jp/graphgolf/2015/candar15/graphgolf2015-mizuno.pdf

Method 1: Approach

- Select Brown graph
  - select odd prime p satisfies $n > p^{2k} + p^k + 1$
  - n becomes discrete value
- Brown graph doesn’t satisfy ODP constraint
  - Add nodes by uniting other graphs
  - Add edges randomly to get the graph regular

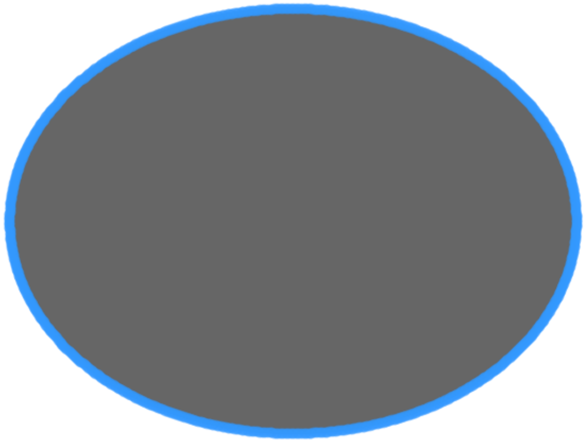
Method 1: Parameters we use

k	p	Order	Degree
1	19	381	20
	31	993	32
	37	1 407	38
	53	2 863	54

In “Graph Golf 2016”, $(n, d) = (1\,024{,}32)$, $(1\,560, 40)$, $(3\,250, 57)$ are featured.

- (1 024, 32)-graph: Use $B(31)$
- (1 560, 40)-graph: Use $B(37)$
- (3 250, 57)-graph: Use $B(19), B(53)$

Method 1: (1 024,32)-graph

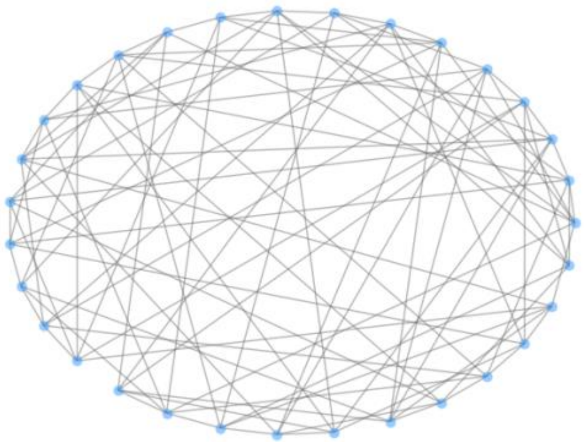


- Brown graph

$n = 993, k = 2$

$d = 32$ (not regular)

$l = 1.967774$



- another graph

$n = 31, k = 3$

$d = 7$ (not regular)

$l = 1.825806$

Method 1: Graph uniting

Brown graphs $B(p)$ are not regular. So, vertices need connecting to another one.

Brute-force approach

- Choose two vertices i, j
- Add an edge between them
- Try to connect all combinations
- Leave the better one.

After that we use 2-opt

Method 1: Results

1 024 nodes, degree 32

Rank	Author	Diam.	ASPL
1	Yawara Ishida & Ryosuke Mizuno	3	1.99784
4	Our team	3	2.01509

defeated

1 560 nodes, degree 40

Rank	Author	Diam.	ASPL
1	Our team	3	2.03823
2	H. Inoue	3	2.18947

3 250 nodes, degree 57

Rank	Author	Diam.	ASPL
1	Our team	3	2.07002
3	H. Inoue	3	2.23220

Roughly work well as we expected

Difficulties in Brown graph

- Degree is not regular
  - p^{2k} vertices have $p+1$, $p+1$ vertices have p
  - We don't have a clue how to unite graphs well

- How to get another graph
  - Random graph with 2-opt
  - Complete graph
  - etc...

We gave up.

Method 2

Node adding method

Cayley graphs provides large (d, k)-graphs in DDP.

- Given m, n, r
  where $r^n \equiv 1 \pmod{m}$, $\gcd(\phi(m), n) > 1$
  $\phi(m)$: Euler's totient function
- Given bouquets
  $B(1, l) = [(a_0, b_0) | (a_1, b_1)(a_2, b_2) \cdots (a_l, b_l)]$ or
  $B(0, l) = [(a_1, b_1)(a_2, b_2) \cdots (a_l, b_l)]$
- Order mn
- Degree $2l + 1$ if using $B(1, l)$, or $2l$ if using $B(0, l)$

Loz, E., & Pineda-Villavicencio, G. (2009). New Benchmarks for Large-Scale Networks with Given Maximum Degree and Diameter. The Computer Journal, 53(7).

Method 2: Example of parameters

m	n	r	bouquets	Order	Degree
46	11	9	$B(0,4)$	506	8
555	18	4	$B(1,3)$	9 990	7
555	18	4	$B(1,5)$	9 990	11
555	18	16	$B(0,10)$	9 990	20
4 165	24	19	$B(0,10)$	99 960	20

In "Graph Golf 2016", $(n, d) = (512{,}8)$, $(10\ 000, 7)$, $(10\ 000, 11)$, $(10\ 000, 20)$, $(100\ 000, 20)$ are featured.

Approach

– Make Cayley graph as a base

– Adding nodes/edges

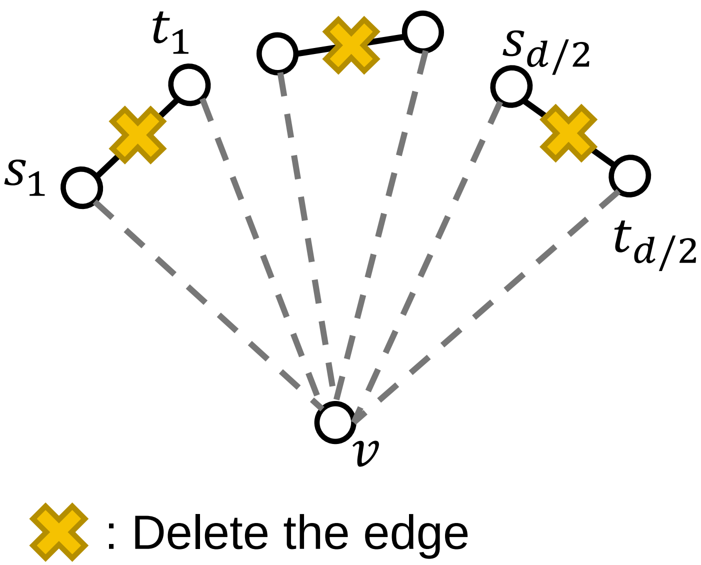
– Turn into desired order/degree graph

Method 2: How to add nodes?

- Add node one by one
- Add edges
  - Then check that the diameter has not increased
- Two procedures
  - Depending on whether order is even or odd

Method 2: Node adding

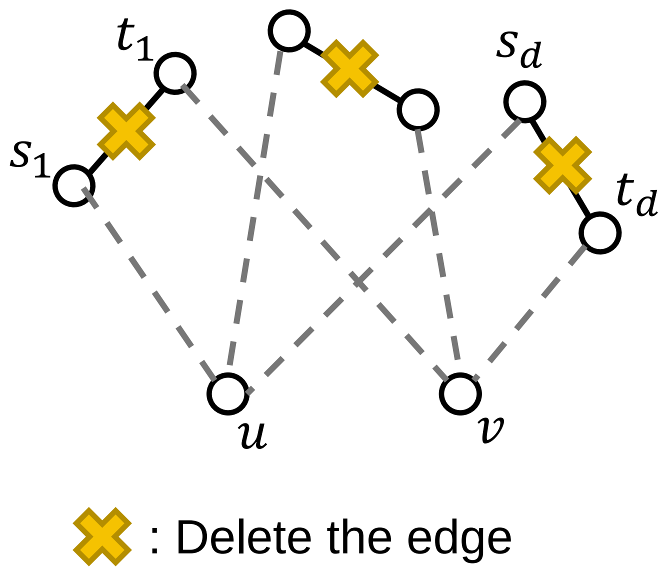
If d is even



1. Select $d/2$ edges which are not adjacent to each other
2. Delete edges selected in 1.
3. Add edges between v and s, t

Method 2: Node adding

If d is odd



1. Select d edges which is not adjacent to each other
2. Delete edges selected in 1.
3. Add edges between u, v and s, t

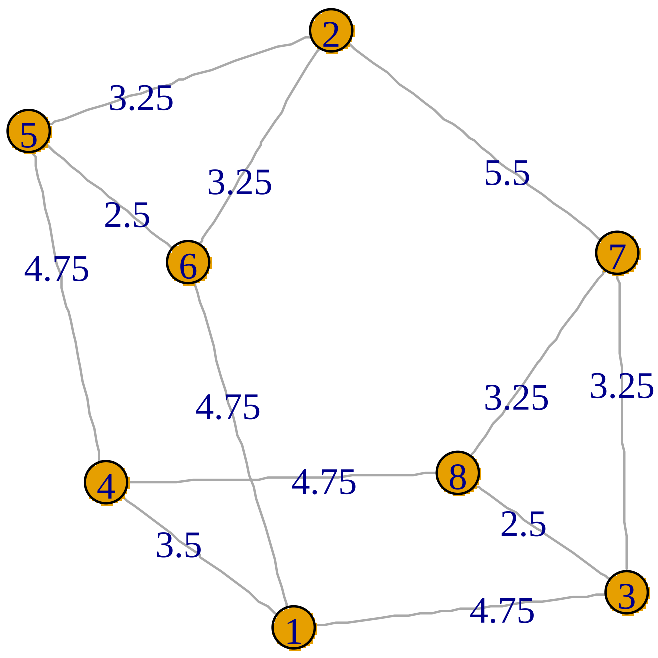
Method 2: How we select edges?

Try not to give a bad effect as possible to diameter/ASPL

Use edge betweenness as an indicator

Edge betweenness

Shows many shortest paths are passing along with the edge



For example (right figure)

original graph: $k = 3$, $l = 1.64$

If deleting edge “2—7”, $k = 4$, $l = 1.89$

If deleting edge “5—6”, $k = 3$, $l = 1.67$

(100 000, 20)-graph: Calculating edge betweenness is time-consuming.
So, we select edges randomly

Ulrik Brandes. (2008). On Variants of Shortest-Path Betweenness Centrality and their Generic Computation. Social Networks, 30(2):136-145.

Method 2: Results

512 nodes, degree 8

Rank	Author	Diam.	ASPL
1	Our team	4	3.11138
4	H. Inoue	4	3.13453

We use 2-opt.
Raw: $k = 4$, $l = 3.1135$

1 024 nodes, degree 8

Rank	Author	Diam.	ASPL
1	Our team	5	3.50538
2	H. Inoue	5	3.51582

Without using 2-opt

100 000 nodes, degree 20

Rank	Author	Diam.	ASPL
1	Our team	5	4.13341
2	H. Inoue	5	4.13621

Without using 2-opt

Method 2: Results

10 000 nodes, degree 7

Rank	Author	Diam.	ASPL
1	Our team	7	5.05994
3	H. Inoue	7	5.07192

Without using 2-opt

10 000 nodes, degree 11

Rank	Author	Diam.	ASPL
1	Our team	5	4.10656
3	H. Inoue	5	4.10802

Without using 2-opt

10 000 nodes, degree 20

Rank	Author	Diam.	ASPL
1	Our team	4	3.37597
3	H. Inoue	4	3.37659

Without using 2-opt

Method 2: Open questions

- How we select m, n, r, bouquets
  - Affect diameter/ASPL of results graph
  - There are many combinations of these pairs

- Applicability of edge betweenness
  - Computational time grows
  - How much it effects to result

Applicability of edge betweenness

- Compare ASPL

Make (512, 8)-graphs for 100 times
They are $k = 4$.

  - Random

min: 3.1161 median: 3.1172 max: 3.1187

  - Use edge betweenness

min: 3.1159 median: 3.1168 max: 3.1180

ASPL became small but
it is hard to say significant effect.

Problems of edge betweenness

It shows how much that edge contributes to shortest paths

- Even though edge betweenness is low, there are shortest paths passing along.
- If we delete that edges, how it affects later is unclear.

Brown graphs or Cayley graphs

- Brown graphs
  - Have advantage ($k = 2$)
  - Not easy to get desired order/degree
- Cayley graphs
  - Can change degree
  - Easy to add nodes/edges
  - Need to choose m, n, r, bouquets

Conclusion

Conclusion

- "Graph Golf"
  - We submitted eleven graphs, and won widest improvement award by using new approach
- Problems on 2-opt search
  - Make random graph and do 2-opt search
  - Execution time grows very rapidly
- Our Approach
  - Find ODP solutions from DDP solutions
  - Use Brown or Cayley graph as a base
  - Turn into desired order/degree graph

References

- Bachratý, M., & Širáň, J. (2014). Polarity graphs revisited. ARS MATHEMATICA CONTEMPORANEA, 8(1).
- Loz, E., & Pineda-Villavicencio, G. (2009). New Benchmarks for Large-Scale Networks with Given Maximum Degree and Diameter. The Computer Journal, 53(7).
- Ulrik Brandes. (2008). On Variants of Shortest-Path Betweenness Centrality and their Generic Computation. Social Networks, 30(2):136-145.