Constructing Large-scale Low-latency Network from Small Optimal Networks

Ryosuke MIZUNO
Department of Physics
Kyoto University

Yawara ISHIDA
Research Institute for
Mathematical Science
Kyoto University

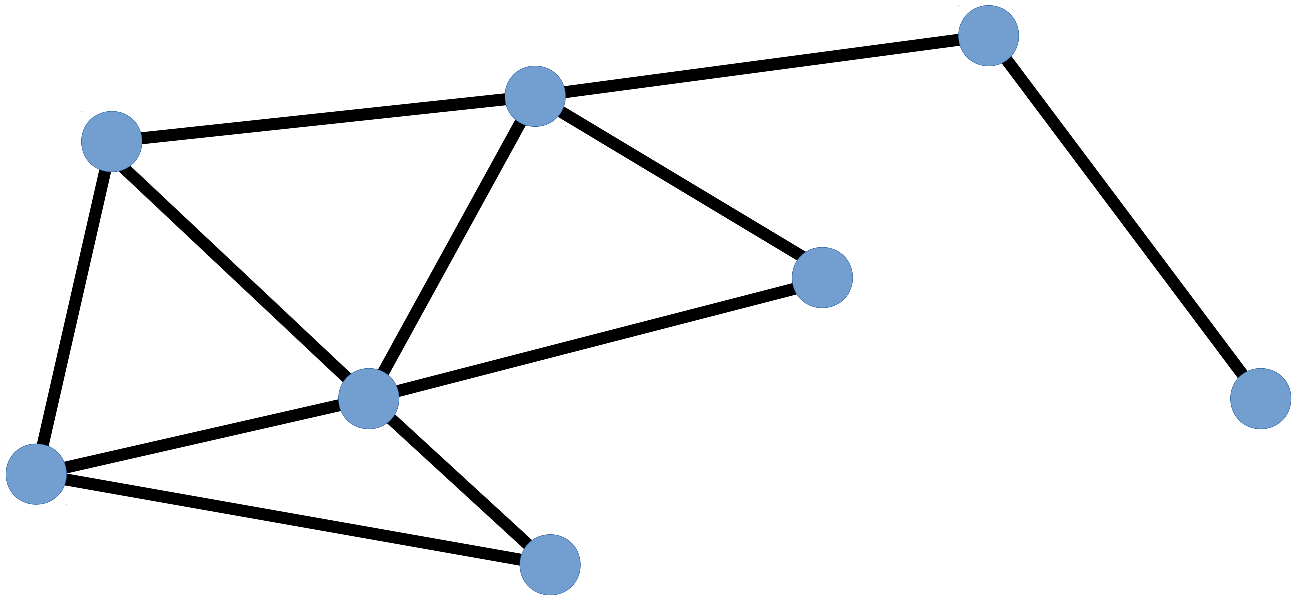
Outline

(1) The Order-Degree Problem (ODP)

(2) Constructions of optimal graphs of the ODP

(3) Summary

Remind definitions



Order = (Number of vertices) = 8

Maximum Degree = (The maximum number of adjacent vertices to a vertex)

= 5

Diameter = (The maximum shortest distance of all pairs of vertices) = 4

The Order-Degree Problem (ODP)

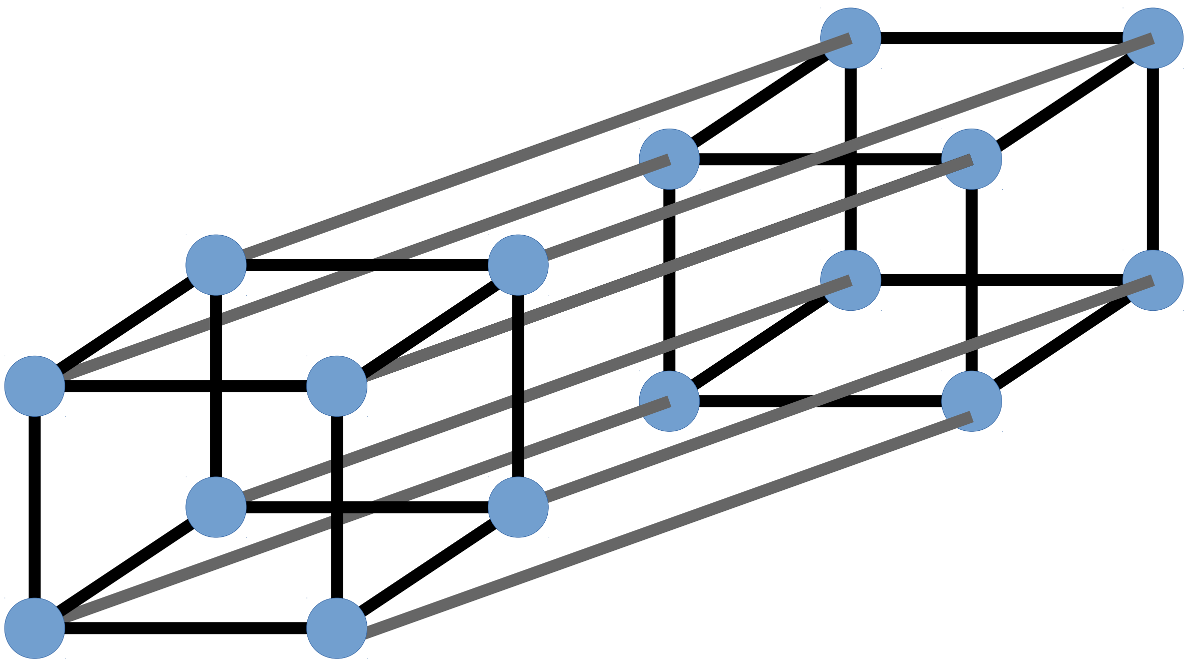
Find graphs with the smallest diameter among graphs of given order and maximum degree.

In an optimal graph for ODP, the distance of each pair of vertices is relatively short.

We could apply ODP to designing of low-latency networks.

Let's consider the ODP for Order =16 and Degree = 4

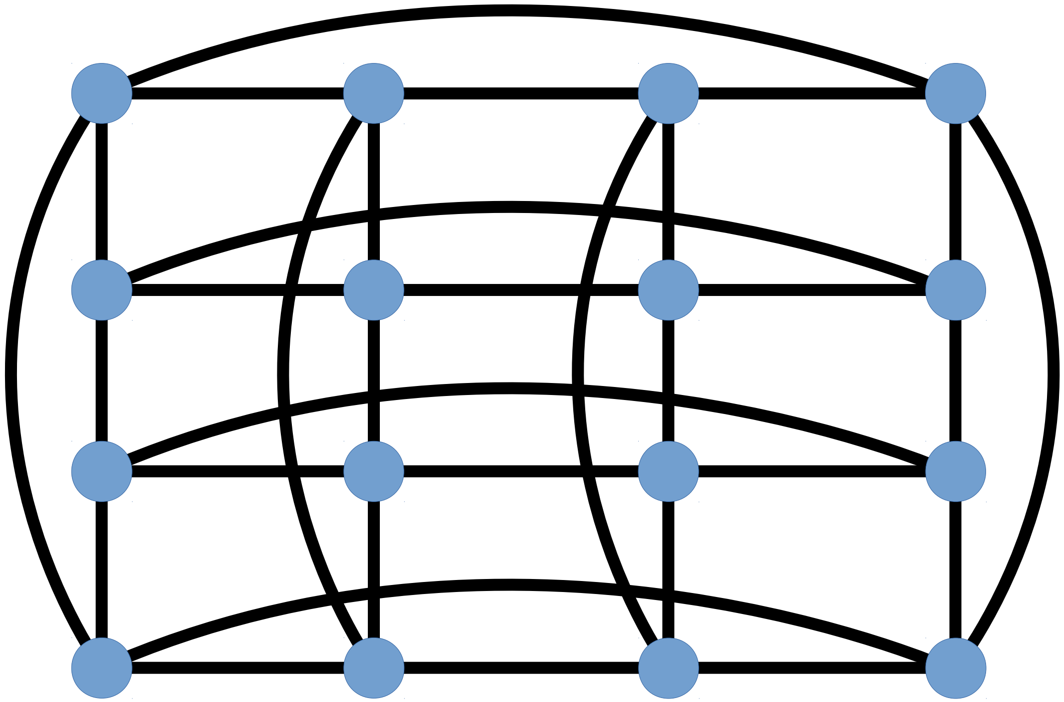
4-dimensional hypercube graph



Diameter = 4
This is not optimal

Let's consider the ODP for Order =16 and Degree = 4

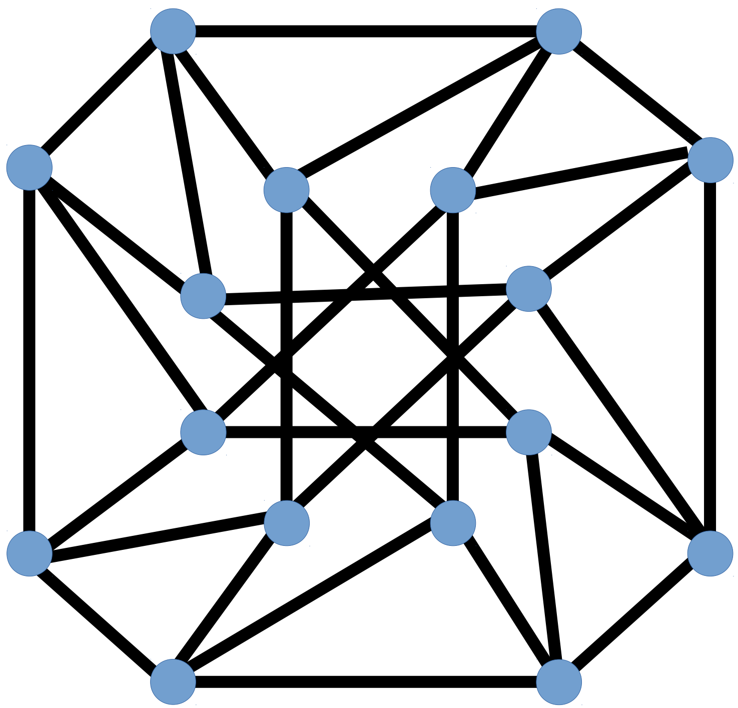
2-dimensional torus grid graph



Diameter = 4
This is not optimal

Let's consider the ODP for Order =16 and Degree = 4

An example of optimal graph (*not unique in general*)



Diameter = 3

It is relatively easy to find optimal graphs of ODP for small order and maximum degree.

Q. How can we find optimal graphs of ODP for large order and maximum degree?

How to find optimal graphs of the ODP for large order and degree

Optimization	Direct construction
・We can try to construct the graph for any pair of order and degree ・But it is difficult to execute calculation for very large order (~1000000) ・Using computer	・We can try to construct the graph for specific pairs of order and degree only ・We can obtain graphs of very large order

How to find optimal graphs of the ODP for large order and degree

We focus on

Optimization	Direct construction
・We can try to construct the graph for any pair of order and degree ・But it is difficult to execute calculation for very large order (~1000000) ・Using computer	・We can try to construct the graph for specific pairs of order and degree only ・We can obtain graphs of very large order

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How to construct a graph

ex.

(1) Browns construction

(2) Generalized Browns construction

(3) Using algebraic structures

(4) Taking products of optimal graphs of ODP

- Cartesian product
- Strong product
- Multiple star product

(1) Cartesian product $G \square H$

(2) Strong product $G \boxtimes H$

(3) Multiple star product $G *_{\varphi_L} H$

・By taking a product of two graphs G and H, we can obtain a new larger graph

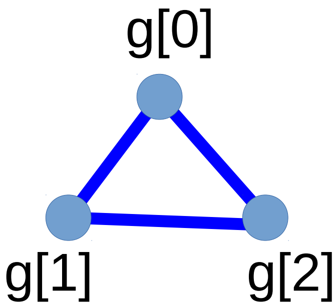
・The cartesian product or strong product of optimal graphs for ODP is not optimal in general but quasi-optimal for ODP (Diameter is relatively small).

・By using the multiple star product, we obtained infinite series of new optimal graphs of ODP.

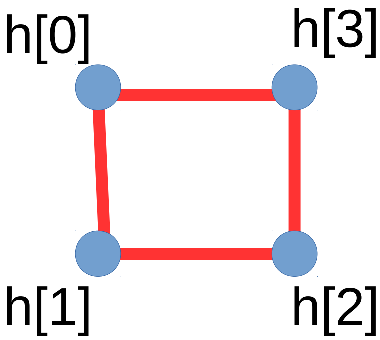
(1) Cartesian product

Step1. Prepare two graphs G and H

G :

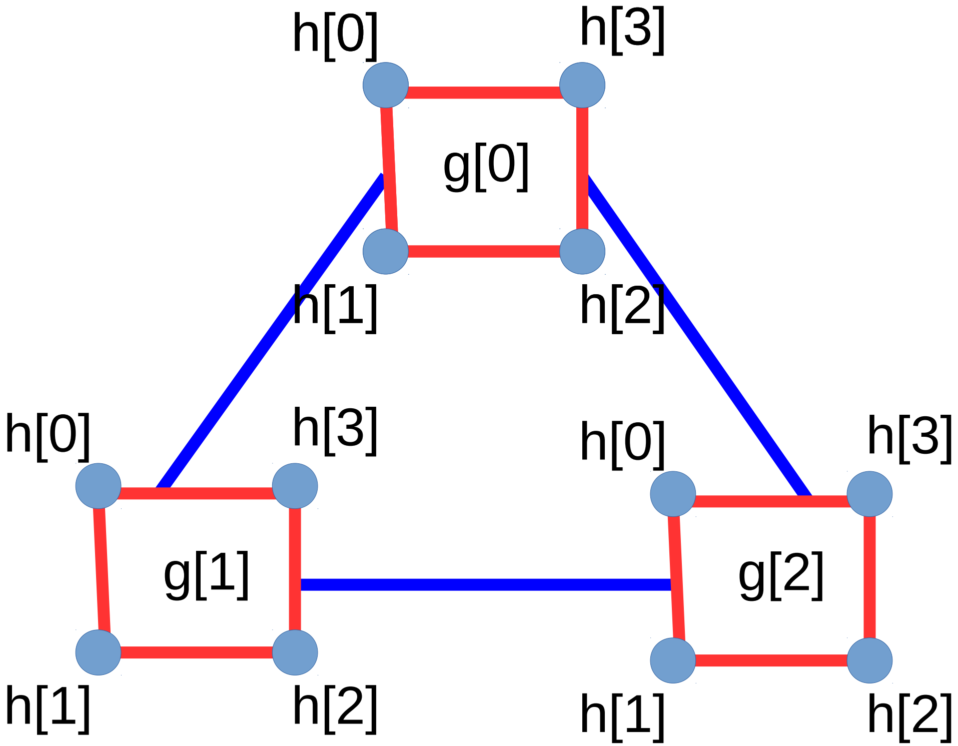


H :



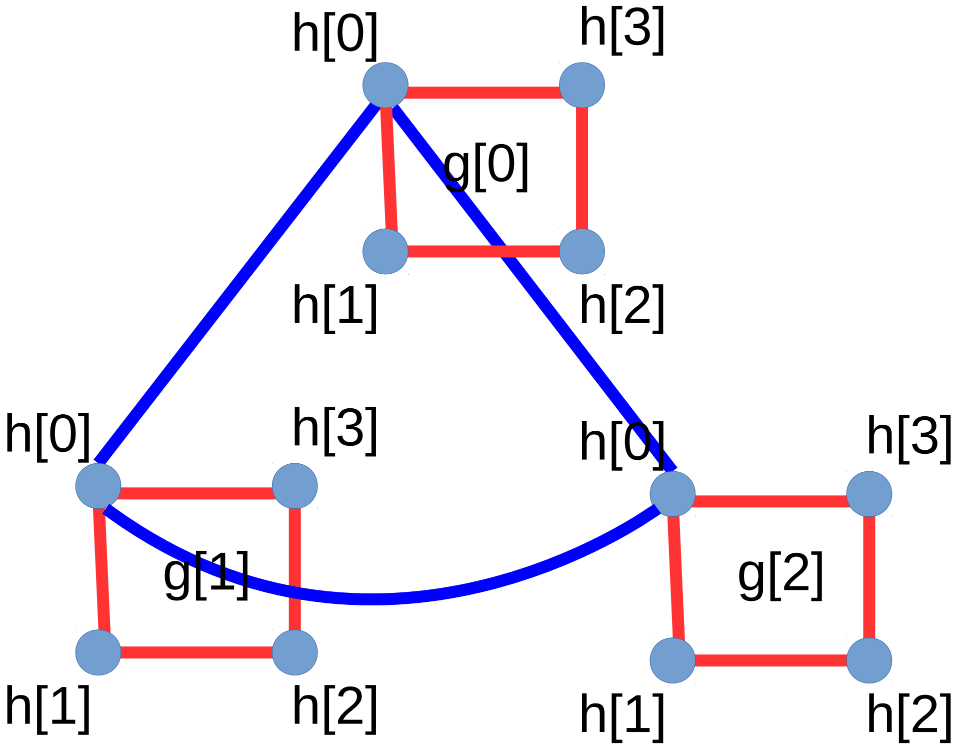
(1) Cartesian product

Step2.
Put a copy of H on each vertex of G



(1) Cartesian product

Step3.
Put a copy of G on each same name vertex of H



(1) Cartesian product

Step3.
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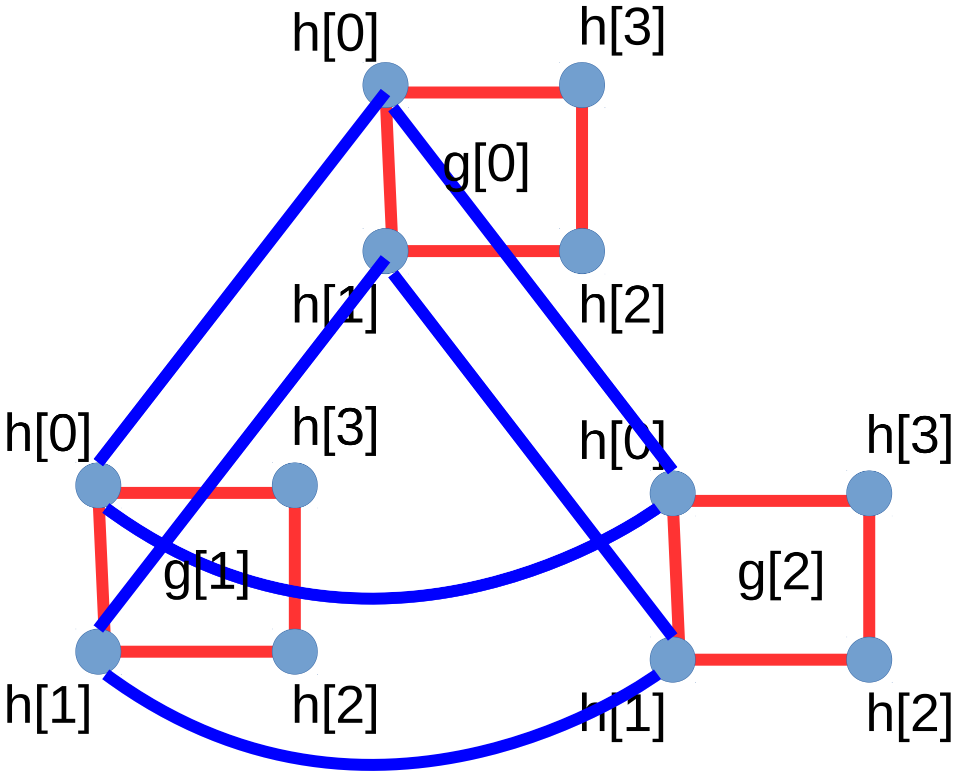

(1) Cartesian product

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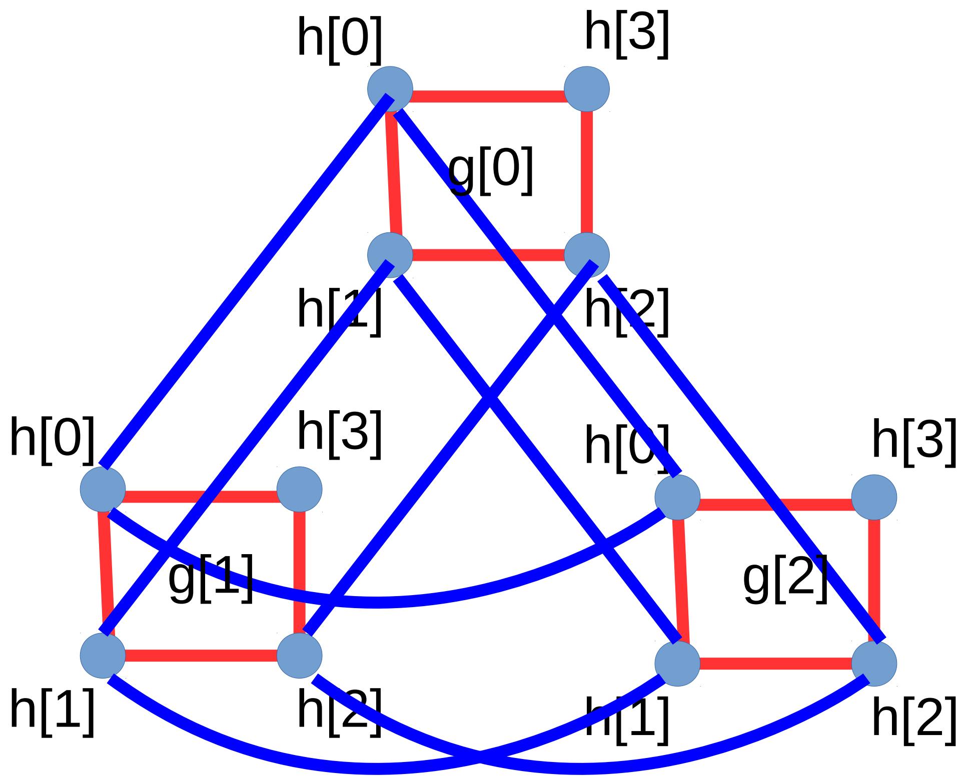

(1) Cartesian product

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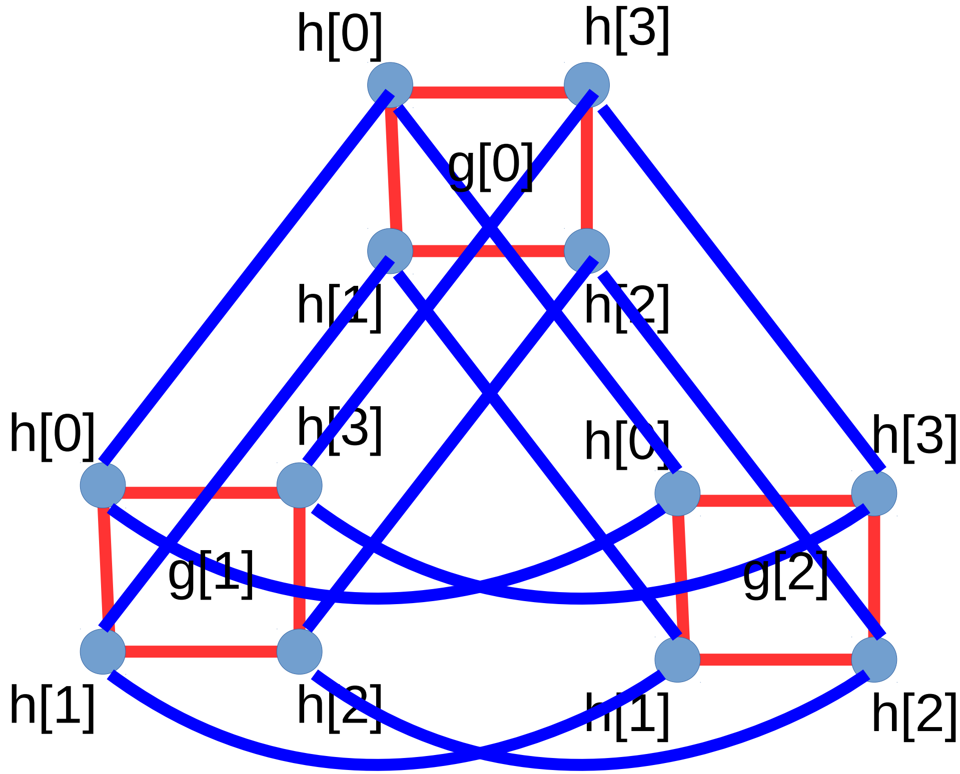
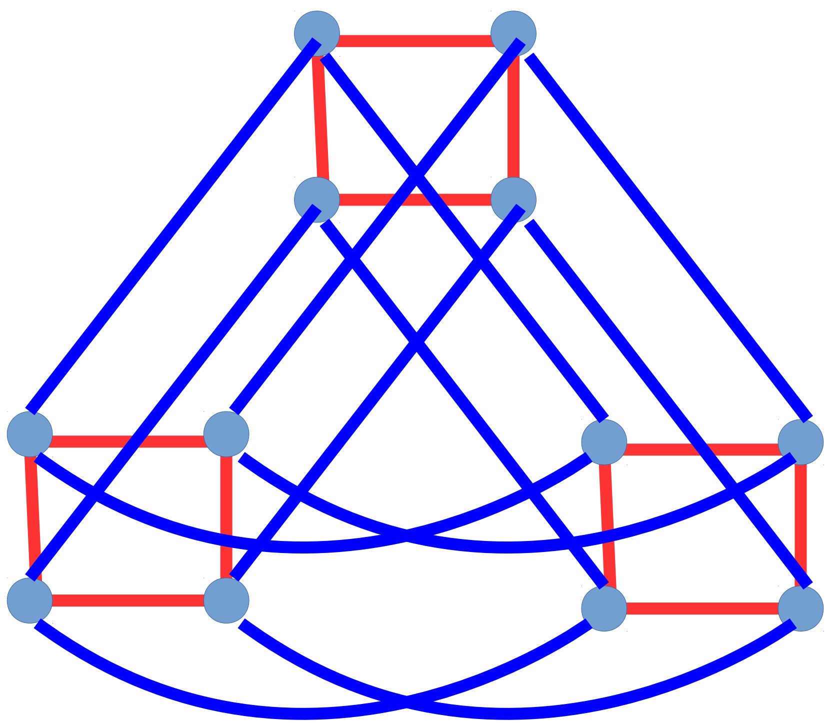


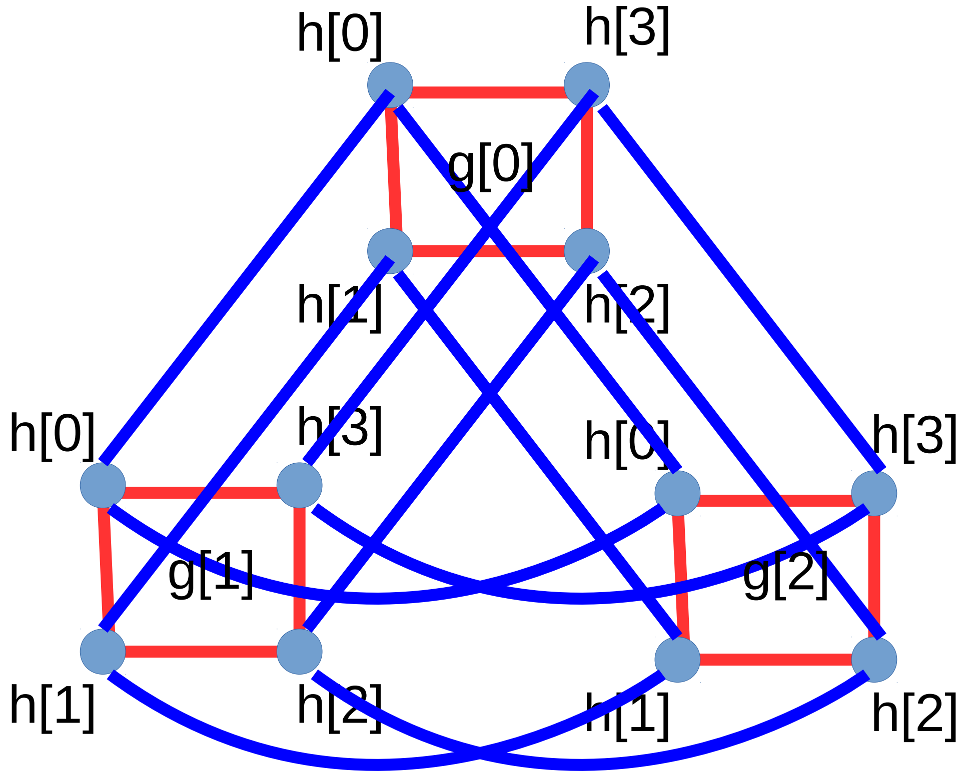
(1) Cartesian product

The cartesian product of G and H



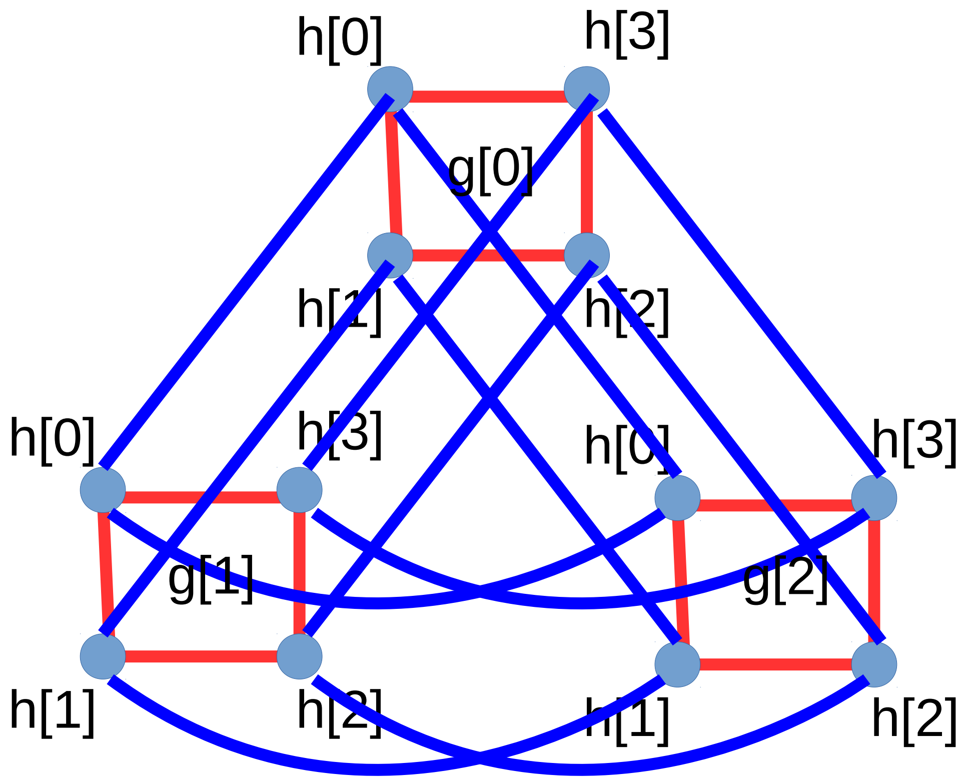
(2) Strong product

Step1, 2 and 3 is same as in the case of cartesian product.



(2) Strong product

Step4.
Draw a line between vertices which are connected by an edge of G and an edge of H



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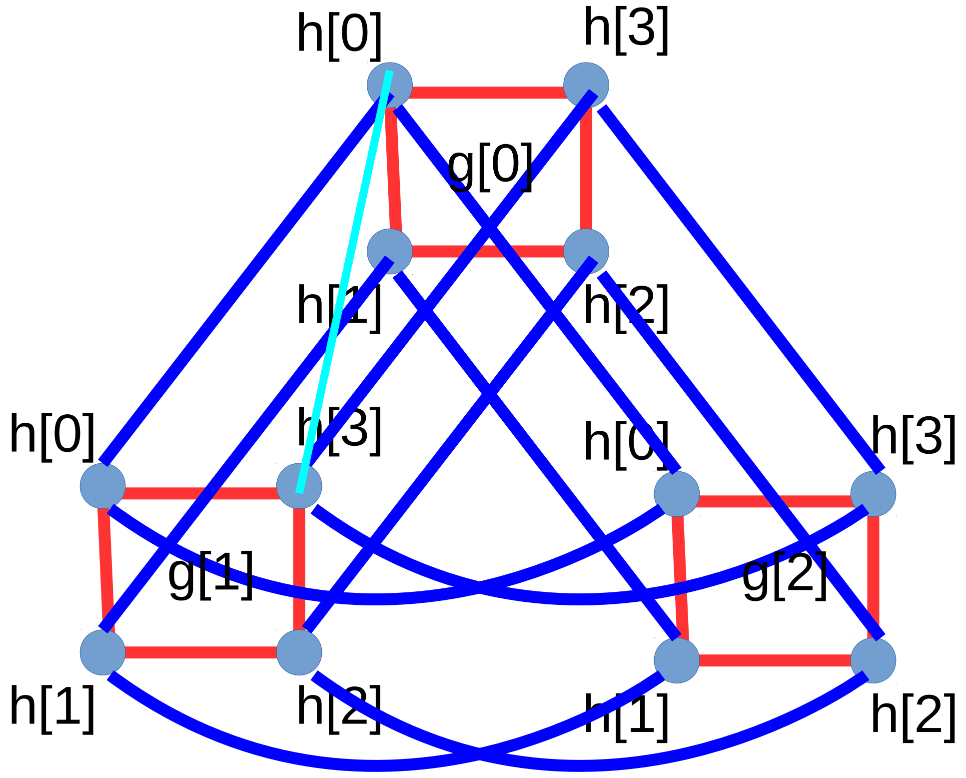
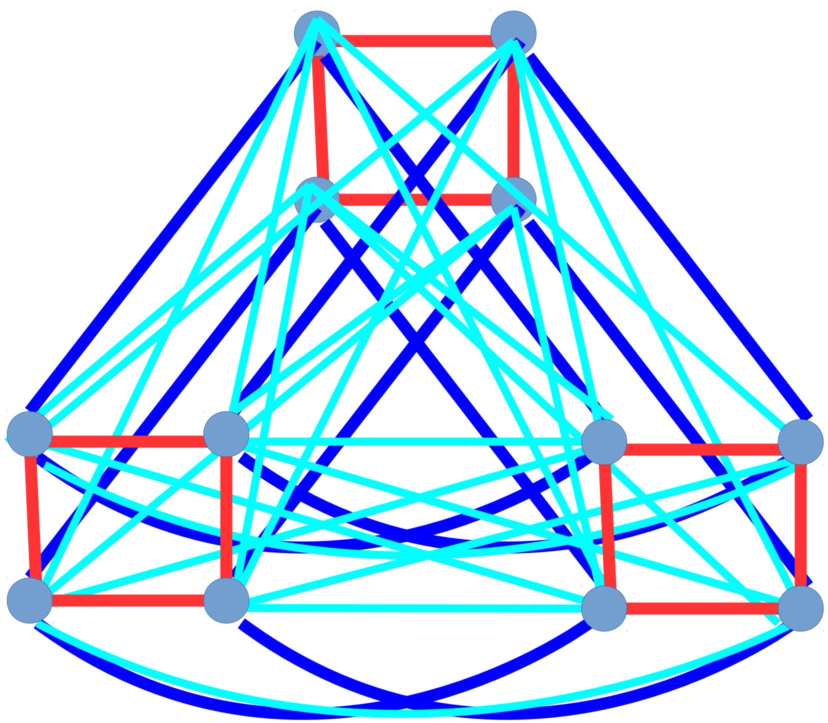


(2) Strong product

The strong product of G and H



The cartesian product or strong product of optimal graphs for ODP is not optimal in general (but quasi-optimal for ODP).

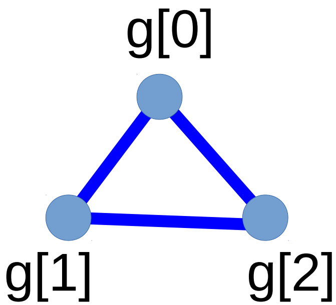
One of the reasons is that we can not consider the inner structure of graphs G and H when we take their product.

On the other hand, by using multiple star product, we can choose the constraction of graphs depending on the inner structure of graphs G and H.

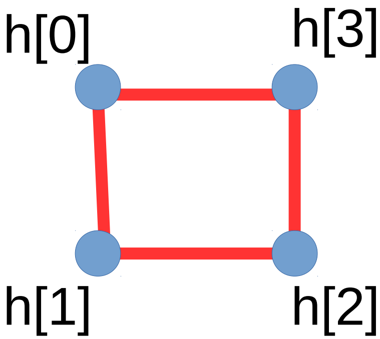
(3) Multiple star product

Step1. Prepare two graphs G and H

G :



H :

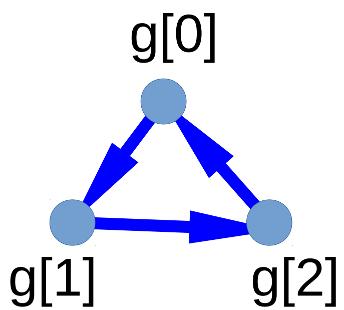


(3) Multiple star product

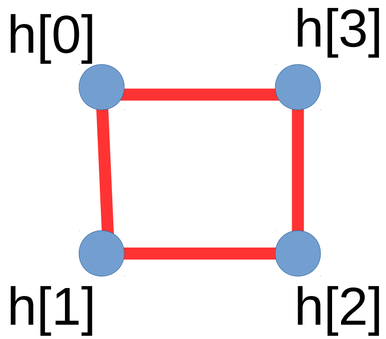
<u>Step2.</u>

- Choose a direction for each edge of G
- Prepare permutations $\varphi_L = (\varphi_1 , \dots , \varphi_n)$ on vertex set of H

G :



H :

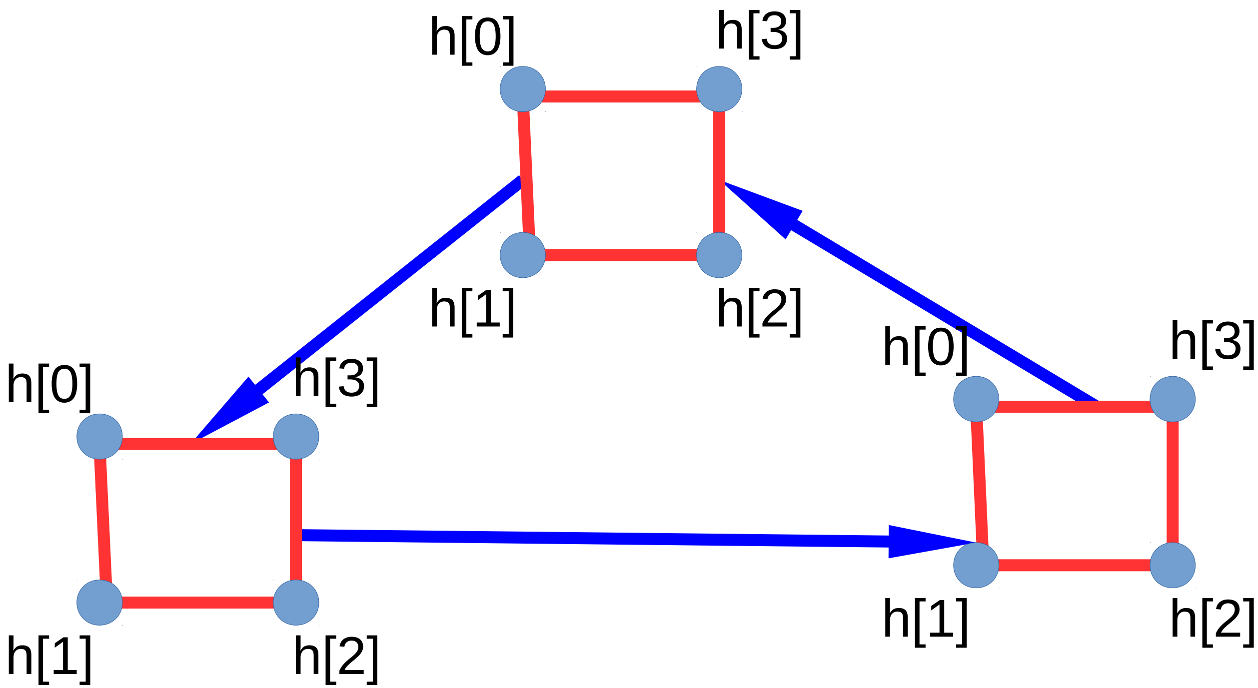


$$\varphi_1 (h[n]) = h[n+1 \bmod 4]$$

$$\varphi_2 (h[n]) = h[n-1 \bmod 4]$$

(3) Multiple star product

Step3.
Put a copy of H on each vertex of G

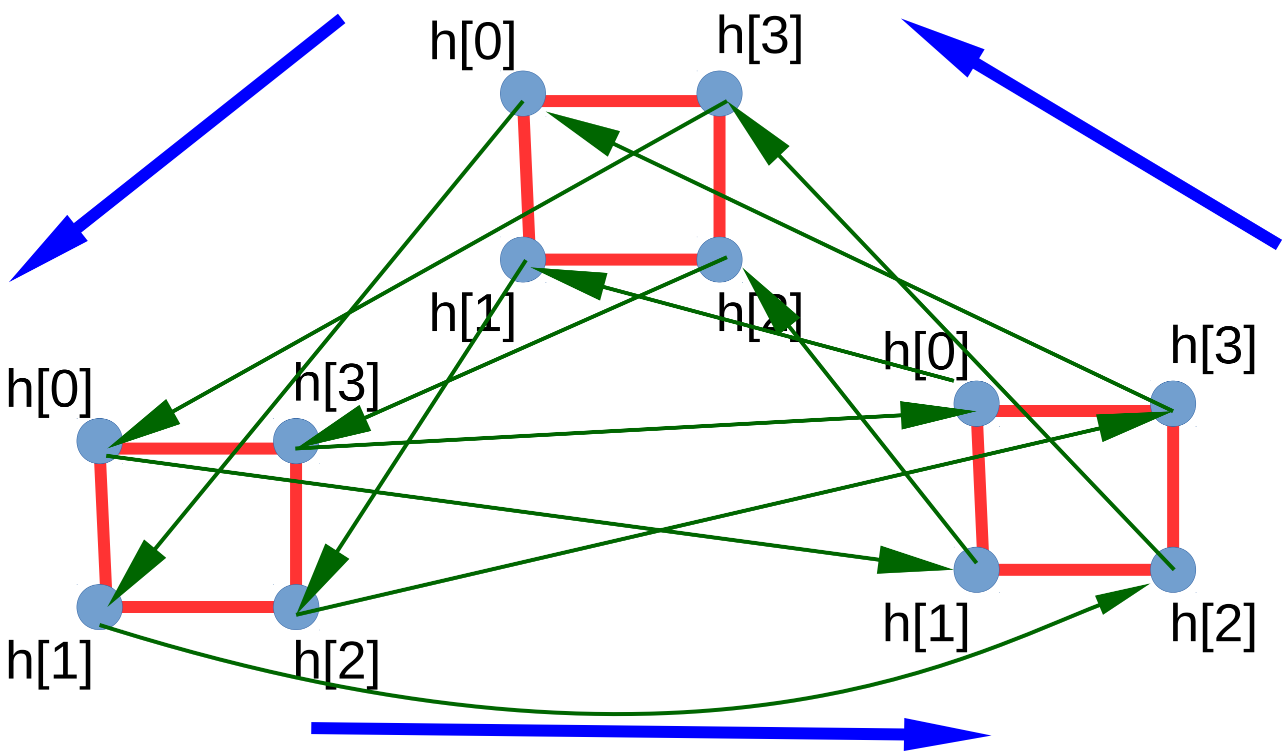


$$\varphi_1(h[n]) = h[n+1 \bmod 4]$$
$$\varphi_2(h[n]) = h[n-1 \bmod 4]$$

(3) Multiple star product

Step4.

Draw allows of the permutation of each vertex of H along directed edge of G



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(3) Multiple star product

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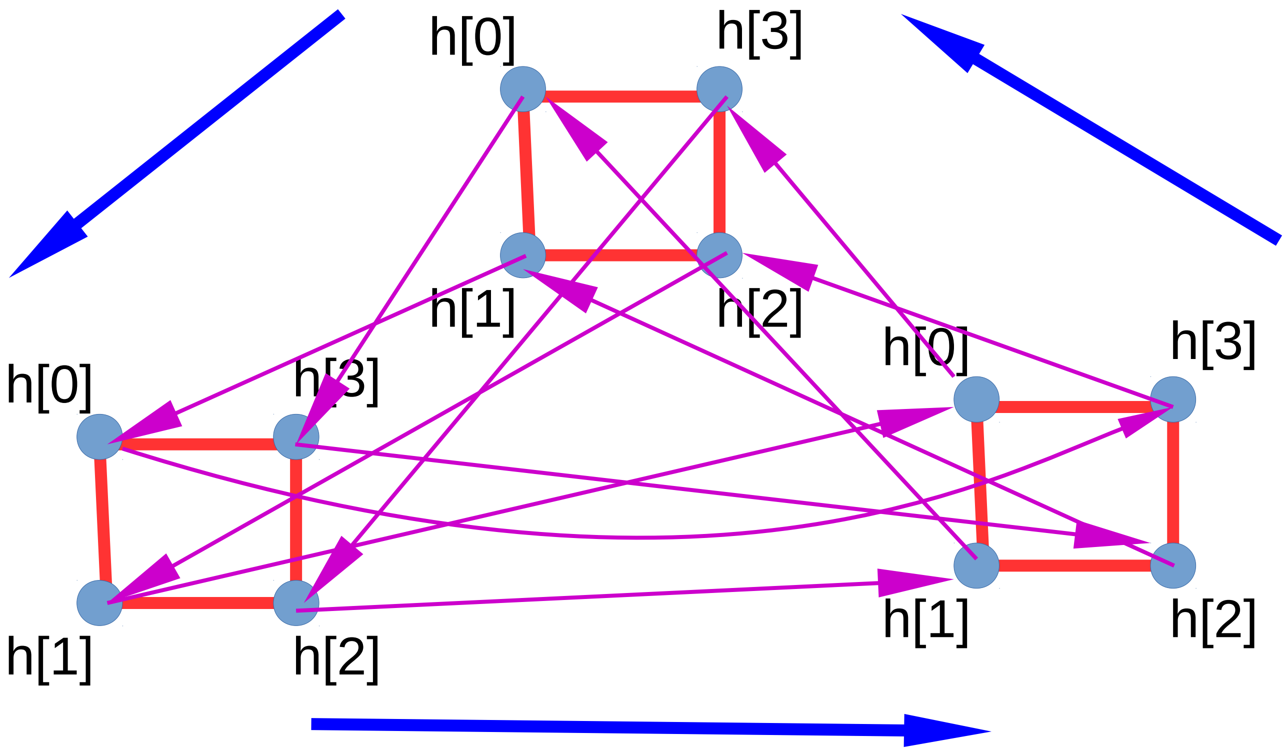

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(3) Multiple star product

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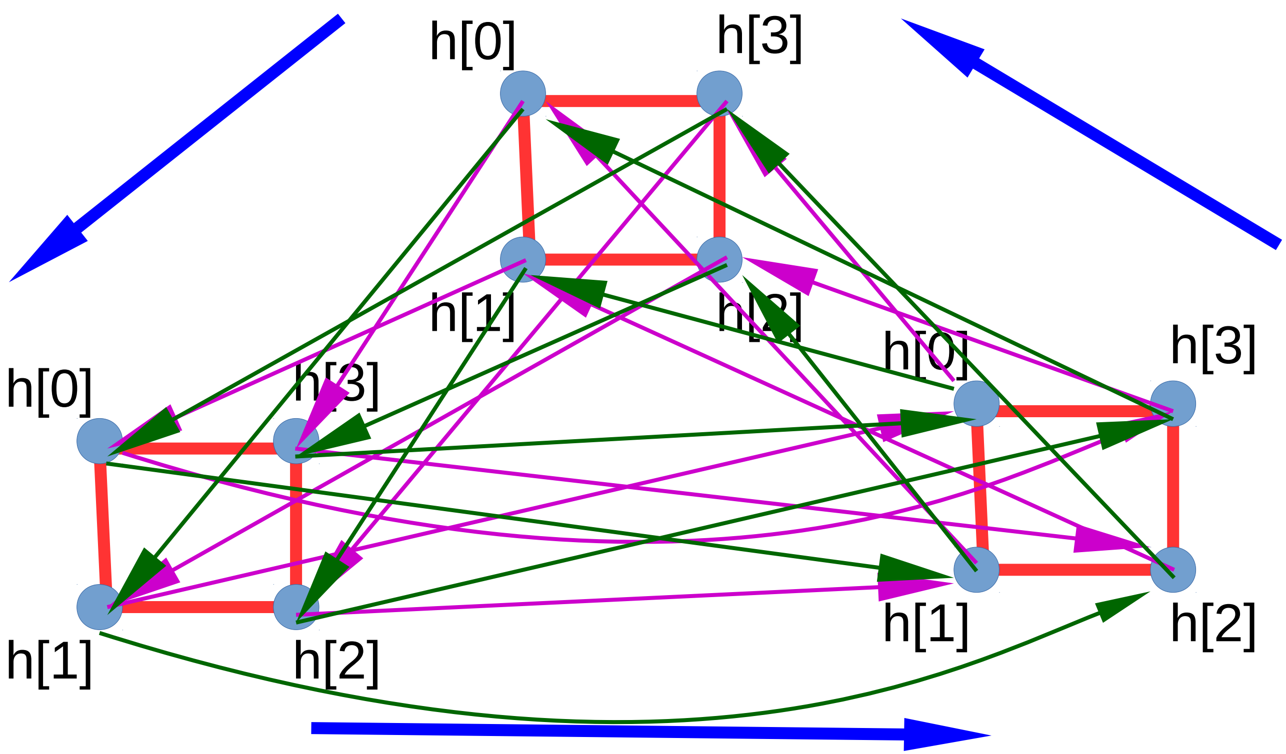


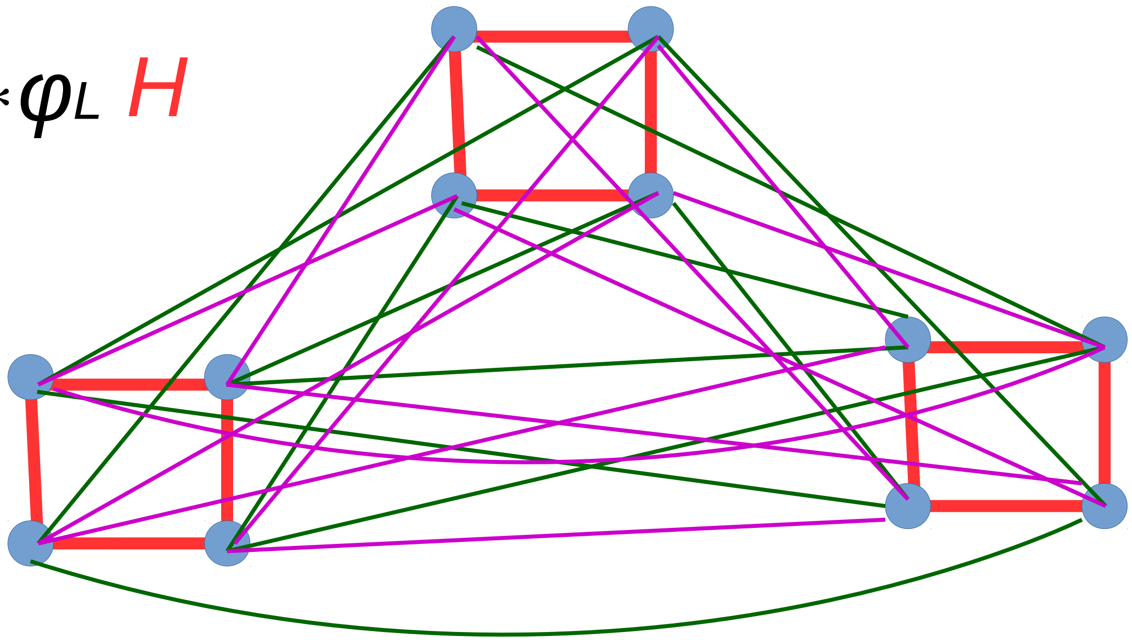
$\varphi_1(h[n]) = h[n+1 \bmod 4]$

$\varphi_2(h[n]) = h[n-1 \bmod 4]$

(3) Multiple star product

Step5.
Replace allows to edges

$G *_{\varphi_L} H$



$\varphi_1(h[n]) = h[n+1 \bmod 4]$

$\varphi_2(h[n]) = h[n-1 \bmod 4]$

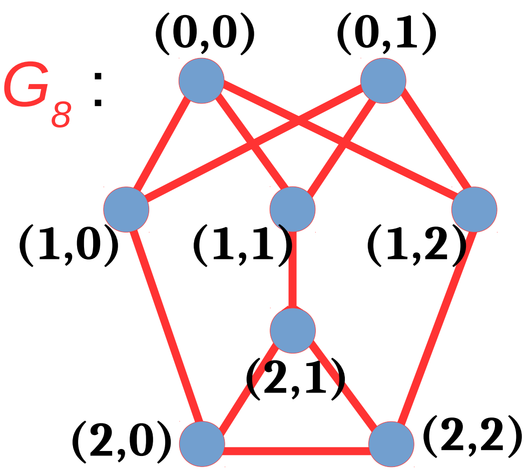
Example $C_k *\varphi\ G_8$

C_k : k-complete graph

・ Vertex set = {0,1,...,k−1}

・ All pairs of vertex are connected each other.

G_8 :

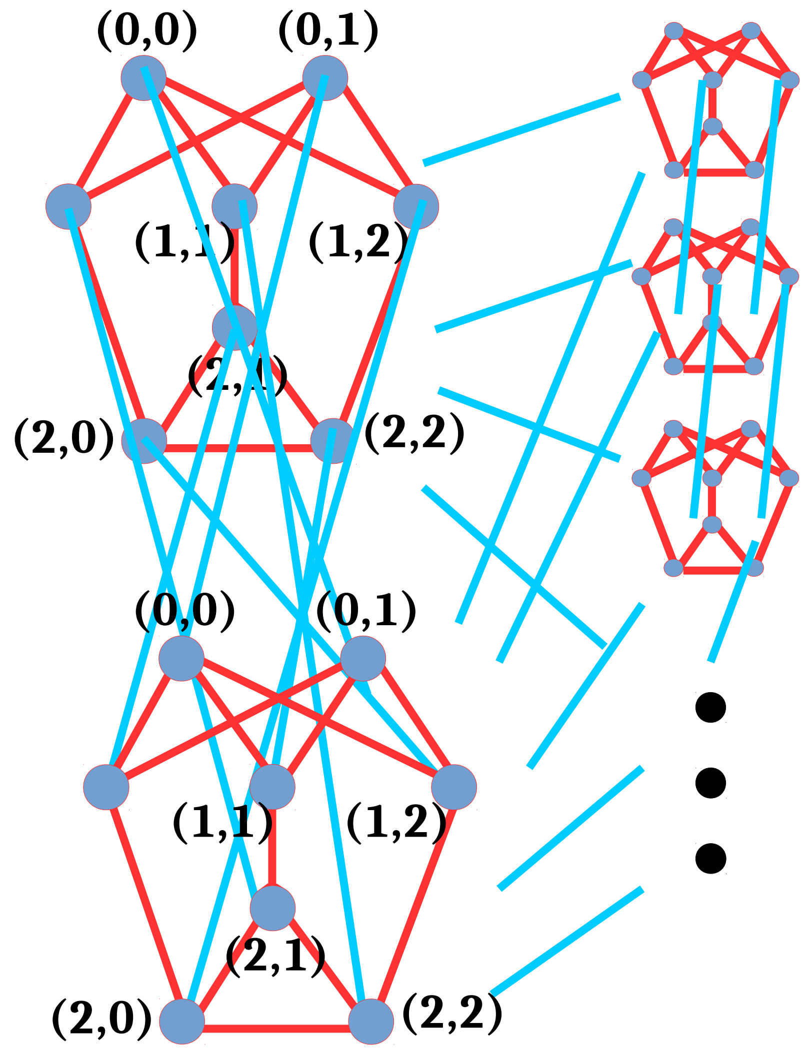


$$\varphi((i,j)) = \begin{cases} (0, 1-j) & (i=0) \\ (2, j+1 \bmod 3) & (i=1) \\ (1, j-1 \bmod 3) & (i=2). \end{cases}$$

Multiple star product

(2) constructions of solutions of ODP

$C_k *\varphi\ G_8$



Example $C_k *\varphi\, G_8$

C_k : k-complete graph

- Vertex set = {0,1,...,k−1}
- All pairs of vertex are connected each other

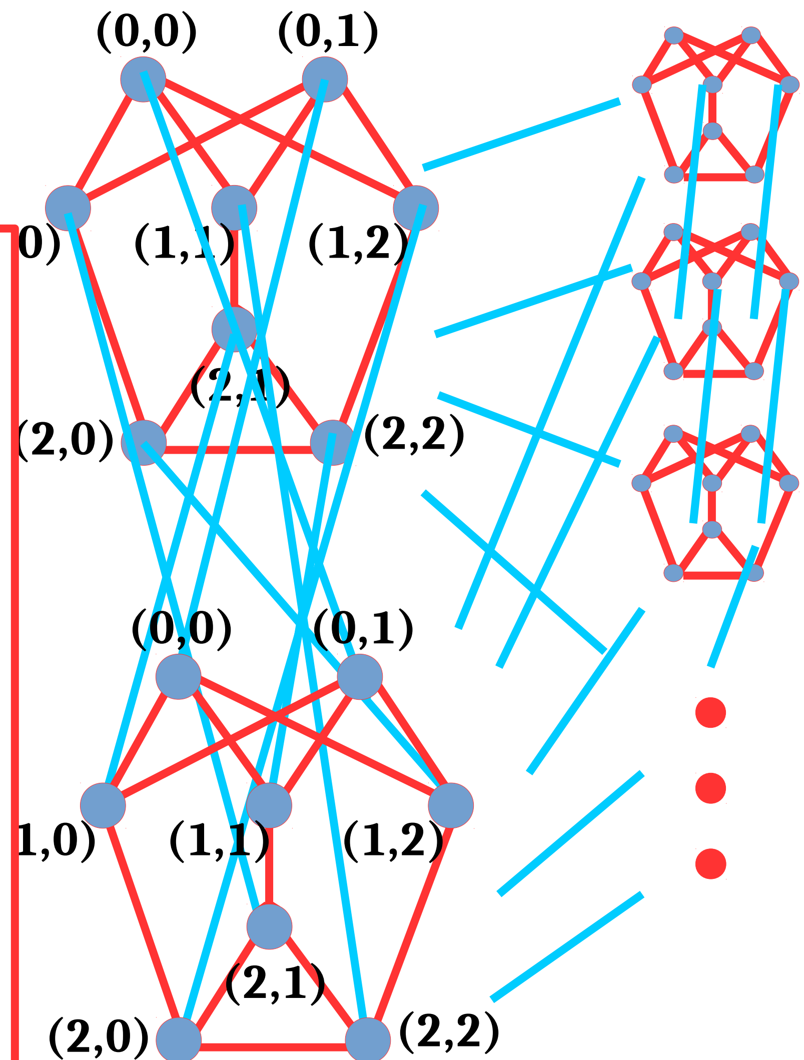
Order = 8k
Degree = k + 2
Diameter = 2 (if k = 2, Diameter = 3)

- *$C_k *\varphi\, G_8$ is an optimal graph of ODP for any k.*
- *The degree of $C_k *\varphi\, G_8$ is the smallest among the graphs of same order and diameter for k = 1,2,3,4,5.*

The connection of vertices is most efficient in this case.

(2) constructions of solutions of ODP

$C_k *\varphi\, G_8$



New solutions. $C_m \ast\psi L\ (C_k \ast\varphi\ G_8)$

The multiple star product of C_m (m-complete graph) and $C_k \ast\varphi\ G_8$

Let $(k,(i,j))$ be the vertex of $C_k \ast\varphi\ G_8$. we define the permutations on the $C_k \ast\varphi\ G_8$, ψ_L, as follows.

$$\psi_1((k,(i,j)))=\begin{cases}(k,(0,0)) & ((i,j)=(0,0))\\(k,(0,1)) & ((i,j)=(1,0))\\(k,(1,0)) & ((i,j)=(1,1))\\(k,(2,1)) & ((i,j)=(1,2))\\(k,(1,1)) & ((i,j)=(0,1))\\(k,(2,2)) & ((i,j)=(2,0))\\(k,(1,2)) & ((i,j)=(2,1))\\(k,(2,0)) & ((i,j)=(2,2))\end{cases}$$

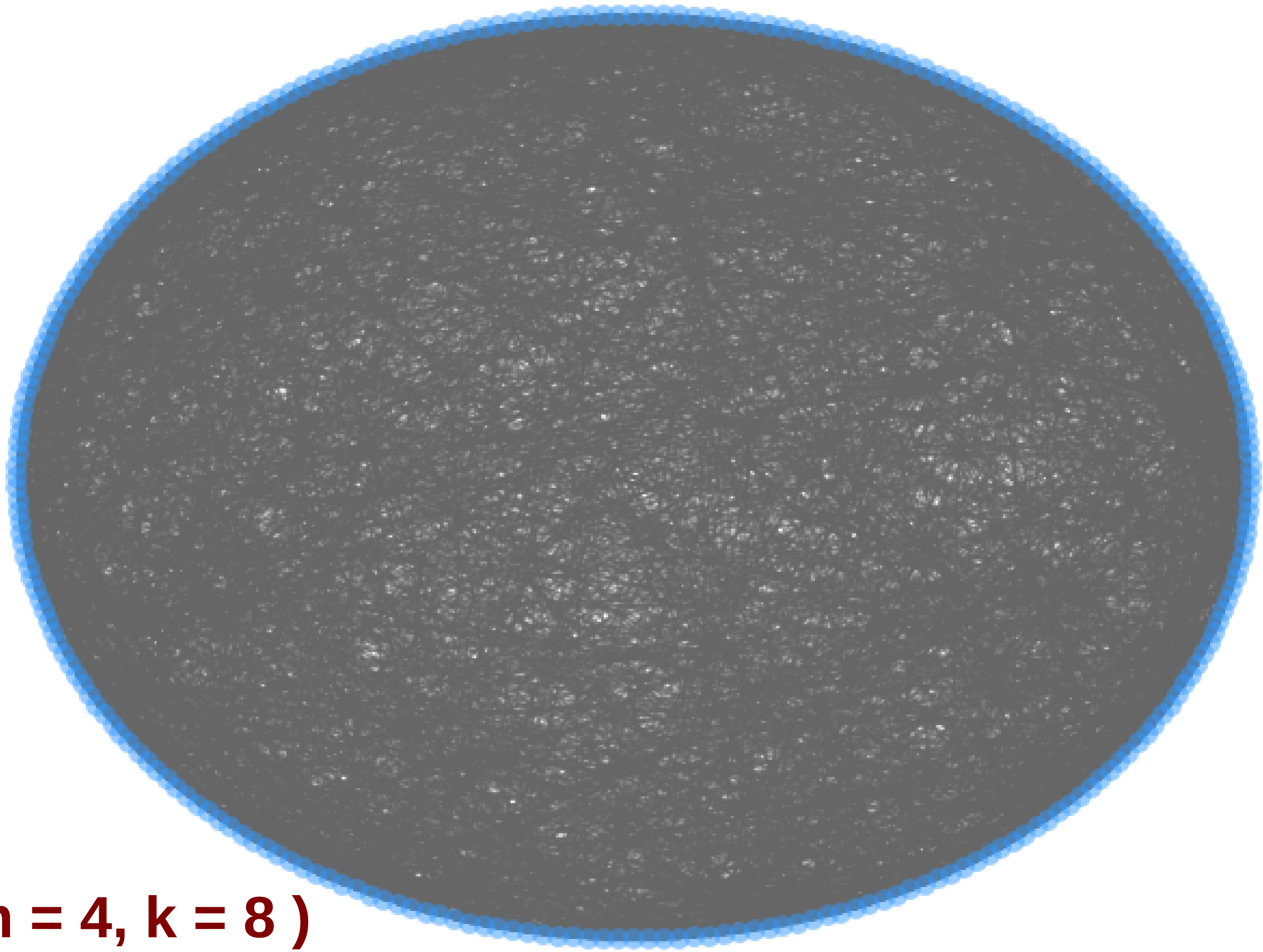
$$\psi_3((k,(i,j)))=\begin{cases}(k,(1,0)) & ((i,j)=(0,0))\\(k,(2,1)) & ((i,j)=(1,0))\\(k,(0,0)) & ((i,j)=(1,1))\\(k,(0,1)) & ((i,j)=(1,2))\\(k,(1,2)) & ((i,j)=(0,1))\\(k,(2,0)) & ((i,j)=(2,0))\\(k,(1,1)) & ((i,j)=(2,1))\\(k,(2,2)) & ((i,j)=(2,2))\end{cases}$$

$$\psi_2((k,(i,j)))=\begin{cases}(k,(0,1)) & ((i,j)=(0,0))\\(k,(1,0)) & ((i,j)=(1,0))\\(k,(2,1)) & ((i,j)=(1,1))\\(k,(0,0)) & ((i,j)=(1,2))\\(k,(2,2)) & ((i,j)=(0,1))\\(k,(1,2)) & ((i,j)=(2,0))\\(k,(2,0)) & ((i,j)=(2,1))\\(k,(1,1)) & ((i,j)=(2,2))\end{cases}$$

$$\psi_4((k,(i,j)))=\begin{cases}(k,(2,1)) & ((i,j)=(0,0))\\(k,(0,0)) & ((i,j)=(1,0))\\(k,(0,1)) & ((i,j)=(1,1))\\(k,(1,0)) & ((i,j)=(1,2))\\(k,(2,0)) & ((i,j)=(0,1))\\(k,(1,1)) & ((i,j)=(2,0))\\(k,(2,2)) & ((i,j)=(2,1))\\(k,(1,2)) & ((i,j)=(2,2))\end{cases}$$

4-multiple star product

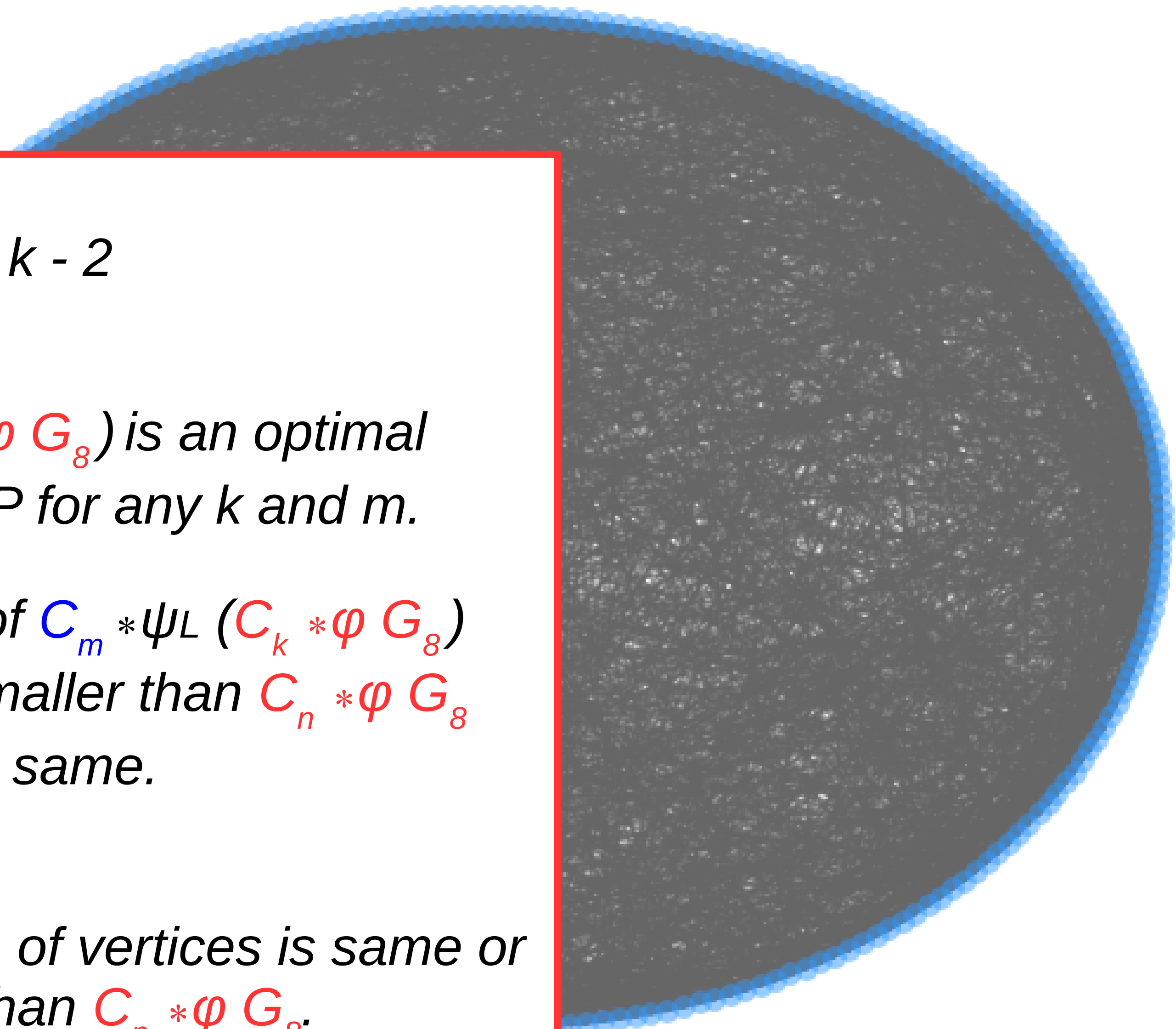
<u>*New solutions.*</u> $C_m * \psi_L (C_k * \varphi G_8)$



(m = 4, k = 8)

(Graph Golf 2015 <http://research.nii.ac.jp/graphgolf/2015/ranking.htm

<u>New solutions.</u> $C_m *_{\psi L} (C_k *_{\varphi} G_8)$



Order = 8km
Degree = 4m + k - 2
Diameter = 2

- *$C_m *_{\psi L} (C_k *_{\varphi} G_8)$ is an optimal graph of ODP for any k and m.*
- *The degree of $C_m *_{\psi L} (C_k *_{\varphi} G_8)$ is same or smaller than $C_n *_{\varphi} G_8$ if the order is same.*

The connection of vertices is same or more efficient than $C_n *_{\varphi} G_8$.

arch.nii.ac.jp/graphgolf/2015/ranking.htm

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(1) The Order-Degree Problem

(2) Constructions of optimal graphs of the ODP

$\boxed{\text{(3) Summary}}$

(3) Summary

- ODP is to find graphs with the smallest diameter among graphs of given order and maximum degree.
- The product of optimal graphs for ODP is not optimal in general but quasi-optimal (Diameter is relatively small).
- By using multiple star product, we can construct a very efficient and small diameter graph. But we have to find good permutation φ_L.
- By using multiple star product, we obtained infinite series $C_m *\psi_L (C_k *\varphi\, G_8)$ of new solutions of ODP.

Graph G

order = O_g

degree = d_g

diameter = D_g

Graph H

order = O_h

degree = d_h

diameter = D_h

Cartesian product

$G \,\square\, H$

order = $O_g \times O_h$

degree = $d_g + d_h$

diameter = $D_g + D_h$

Strong product

$G \boxtimes H$

order = $O_g \times O_h$

degree = $(d_g+1) \times (d_h+1)-1$

diameter = $\max(D_g, D_h)$