ASPL Optimization Approach Using Brown and Cayley Graphs

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Graph Golf 2017

- Order/Degree problem
  - Find a graph with minimum diameter and average shortest path length (ASPL)
  - Given order/degree pairs
- Our results
  - We submitted five best solutions
  - We won General graph widest improvement and Grid graph deepest improvement

Results

1 344 nodes, degree 30

Rank	Author	Diam.	ASPL
1	Our team	3	2.34582
3	Ibuki Kawamata	3	2.36352

4 896 nodes, degree 24

Rank	Author	Diam.	ASPL
1	Our team	4	2.89823
5	Ibuki Kawamata	4	2.90153

9 344 nodes, degree 10

Rank	Author	Diam.	ASPL
1	Our team	5	4.24654
3	Ibuki Kawamata	5	4.25334

Results

88 128 nodes, degree 12

Rank	Author	Diam.	ASPL
1	Our team	6	4.88278
2	Ibuki Kawamata	6	4.88482

98 304 nodes, degree 10

Rank	Author	Diam.	ASPL
1	Our team	7	5.35521
2	Ibuki Kawamata	7	5.35876

16 nodes, degree 3, length 2

Rank	Author	Diam.	ASPL	ASPL gap
1	Our team	3	2.2	0
1	Nakano	3	2.2	0

Approach

Background

- Degree/diameter problem (DDP)
  - Given degree/diameter
  - Find a graph with largest order
- Brown’s construction & Cayley graphs
  - Some DDP solutions are based on them

Approach

1. Brown’s construction

2. Cayley graph as a base

- We used these approach in Graph Golf 2016
  - 2-opt requires a lot of time

Problem

- Brown and Cayley graphs
  - Not always be applied for ODP
  - Use them as a base graph

- To get ASPL optimized graphs
  - Some alternation needed
    1. Unite graphs
    2. Node removing method
    3. Node bisection method

Why we use DDP

Heuristics for diameter 3 graphs

- Create a base graph
  - Targeted diameter only $k = 3$
  - Multiple Petersen graphs are connected
- Greedily add edges
  - To increase the # of pentagons

We described detail in Graph Golf 2015

What we did in Graph Golf 2016

- Problems on heuristics and 2-opt search
  - Make graphs and do 2-opt search
  - Execution time grows very rapidly

- Use Brown and Cayley graphs
  - Won widest improvement award
  - It will be useful if some alternation were applied
  - By adding (or removing) nodes/edges

1. Brown’s construction

Brown's construction

- Described in [1] at Graph Golf 2015
  - It can make a graph $B(q)$

    Order = $q^2 + q + 1$

    Degree = $q + 1$

    Diameter = 2

    q: a prime

[1] R. Mizuno and Y. Ishida, "The construction of a regular graph,"
http://research.nii.ac.jp/graphgolf/2015/candar15/graphgolf2015-mizuno.pdf

Using Brown's construction

- 1344 nodes, degree 30
  - We use $B(25)$ as a base
    - $n = 651,\ d = 26,\ k = 2$
  - Uniting two $B(25)$ and add nodes
    - 2×651 nodes + 42 nodes
  - Add edges randomly
    - Using approach described in [2]
  - Result: $k = 3, l = 2.3466$

[2] T. Matsuzaki *et al.*, "Making smallest-diameter graphs at "Graph Golf", http://research.nii.ac.jp/graphgolf/2016/candar16/graphgolf2016-matsuzaki.pdf

Disadvantage of Brown's construction

- It may be useful for graphs only $k = 3$
- In Graph Golf 2017
  - $n = 32,\ d = 5,\ k = 3$
  - $n = 256,\ d = 18,\ k = 3$
  - $n = 1344,\ d = 30,\ k = 3$
- We couldn't get good base graphs

2. Cayley graph as a base

Cayley graphs

Large (d, k)-graph in DDP (degree/diameter problem) are Cayley graphs [3].

- Given m, n, r
  where $r^n \equiv 1 \pmod{m}$, $\gcd(\phi(m), n) > 1$
  $\phi(m)$: Euler's totient function
- Given bouquets
  $B(1, l) = [(a_0, b_0)|(a_1, b_1)(a_2, b_2) \cdots (a_l, b_l)]$ or
  $B(0, l) = [(a_1, b_1)(a_2, b_2) \cdots (a_l, b_l)]$
- Order mn
- Degree $2l + 1$ if using $B(1, l)$, or $2l$ if using $B(0, l)$

[3] E. Loz and G. Pineda-Villavicencio, "New Benchmarks for Large-Scale Networks with Given Maximum Degree and Diameter", The Computer Journal, 53(7).

Advantage of Cayley graphs

- Fast diameter and ASPL computation
  - Since Cayley graph is vertex-transitive, single-source ASPL l' = all-pair ASPL l.

$$l' = \frac{1}{n-1}\sum_{j=1}^{n-1} d_{0j}$$

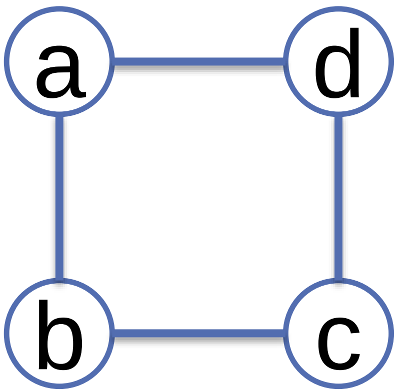
$$l = \frac{1}{n(n-1)}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} d_{ij}$$

where d_{ij} is a path length between node i and j

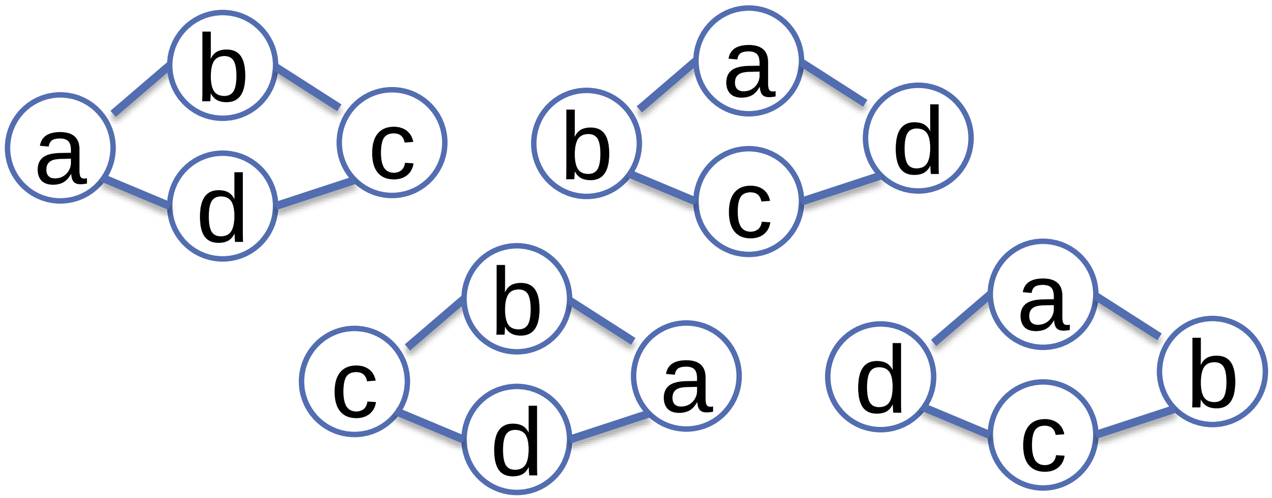
  - Using Dijkstra algorithm, single-source ASPL l' can be calculated faster than all-pair ASPL l.

Vertex-transitive

For example: Square graph



Select a node as root and write like a tree



Distance table

	a	b	c	d
a		1	2	1
b	1		1	2
c	2	1		1
d	1	2	1	

$$l' = \frac{1}{4-1}\underbrace{\sum_{j=1}^{4-1} d_{0j}}_{4} = 1.33$$

$$l = \frac{1}{4(4-1)}\underbrace{\sum_{i=0}^{4-1}\sum_{j=0}^{4-1} d_{ij}}_{16} = 1.33$$

Disadvantages of Cayley graphs

- Not optimal in general

  For example of 256 nodes, degree 18 problem;

  – $k = 3, l = 1.938$: Best of Competition

  – $k = 3, l = 1.984$: Cayley graph

  $m = 8,\ n = 32,\ \ r = 11,$

  B(0,9)=[(1, 17)(2, 25)(3, 28)(4, 21)(4, 29)(5, 13)(7, 11)(7, 16)(7, 28)]

  – $k = 3, l = 2.188$: Random

- For non-arbitrary order $m \times n$

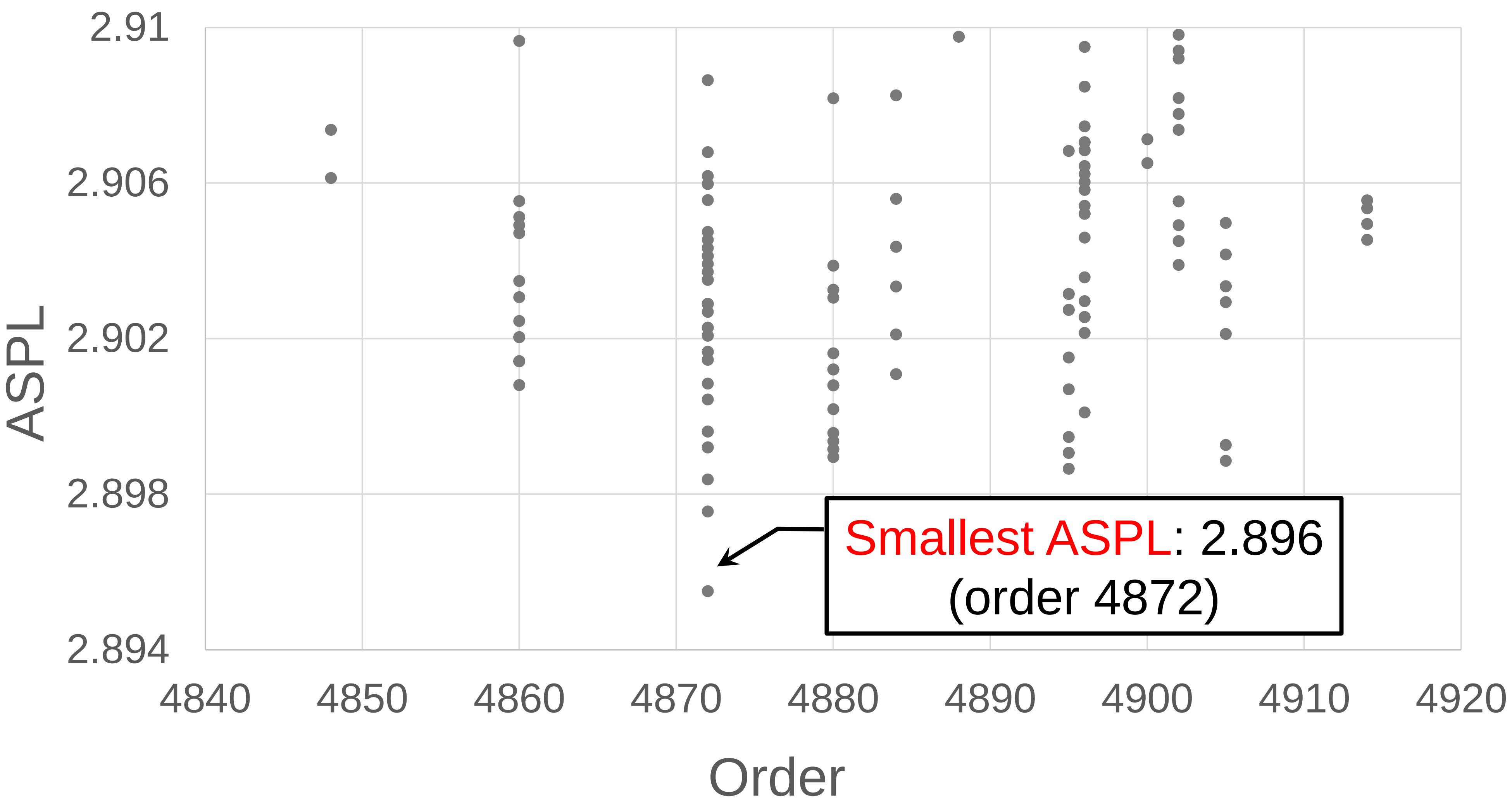
  (m, n, r) should satisfies $r^n \equiv 1 \pmod m$ and $\gcd(\phi(m), n) > 1$.

  – (m, n, r) exist for orders: 250, 252, 253, 256, 258, 260

  – not exist for orders: 251, 254, 255, 257, 259

An example of Cayley graphs

Target: 4896 nodes, degree 24

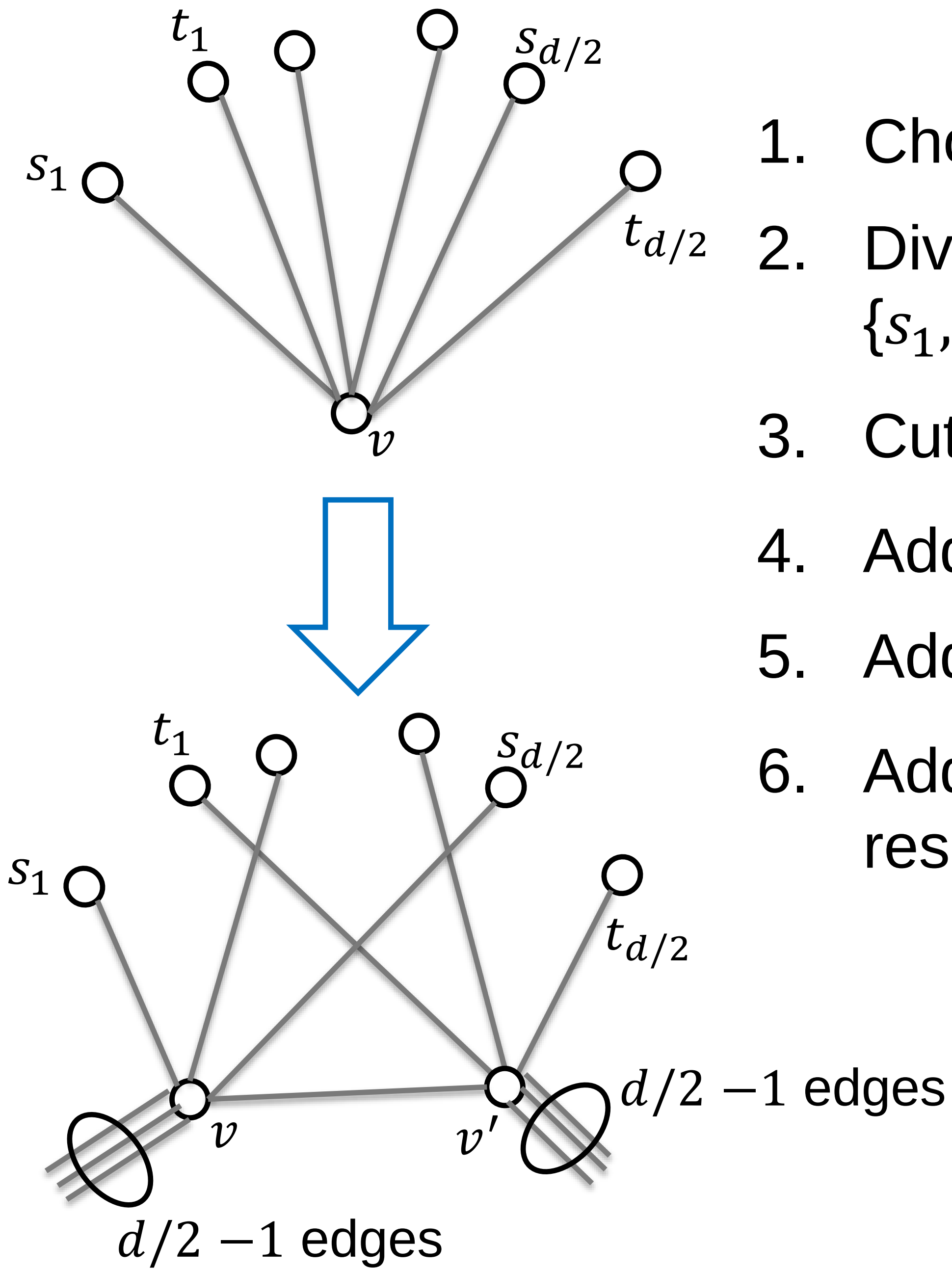


- Select 4872 nodes graph as a base
- Make desired order/degree graph by adding nodes

2. Cayley graph as a base

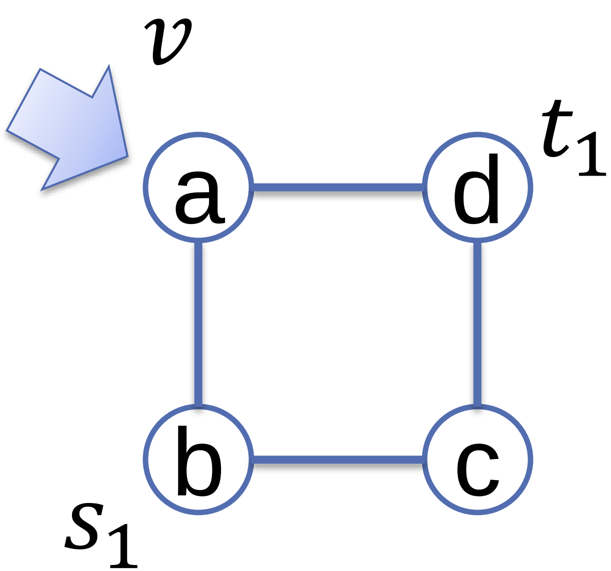
Node bisection method

Node bisection



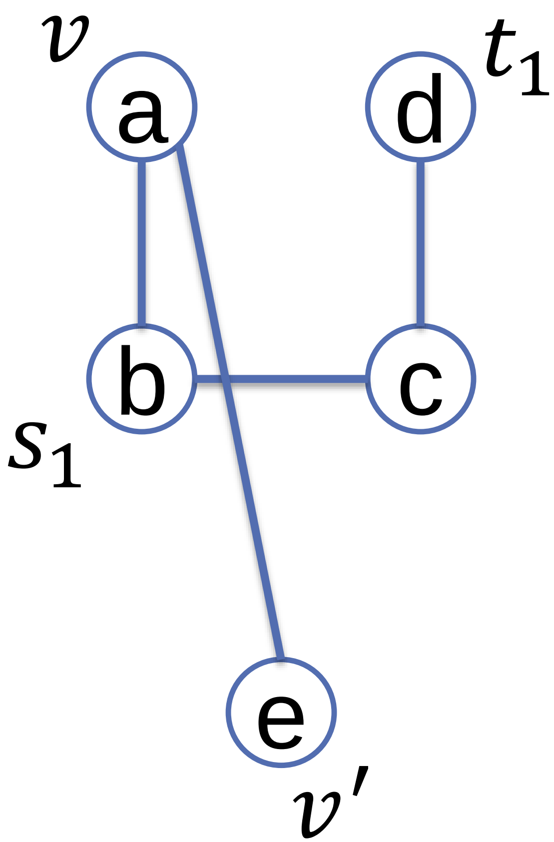
1. Choose a node v
2. Divide d neighbors of v into $\{s_1, \ldots, s_{d/2}\}$, $\{t_1, \ldots, t_{d/2}\}$
3. Cut edges v-t_1 $\cdots$ v-$t_{d/2}$
4. Add bisected node v' and a edge v-v'
5. Add edges v'-t_1 $\cdots$ v'-$t_{d/2}$
6. Add $d/2 - 1$ edges randomly to v and v' respectively (except s, t, v, v')

Node bisection



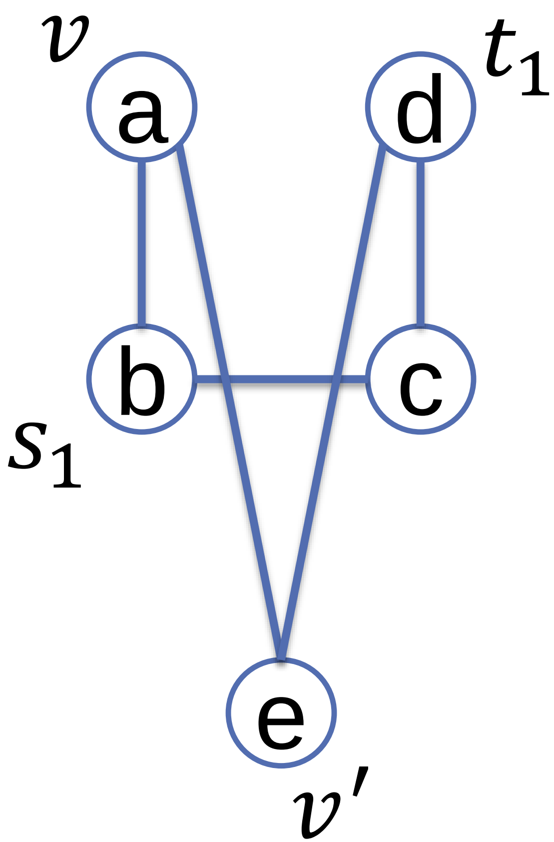
1. Choose a node v
2. Divide d neighbors of v into $\{s_1, \ldots, s_{d/2}\}$, $\{t_1, \ldots, t_{d/2}\}$
3. Cut edges v-t_1 $\cdots$ v-$t_{d/2}$
4. Add bisected node v' and a edge v-v'
5. Add edges v'-t_1 $\cdots$ v'-$t_{d/2}$
6. Add $d/2 - 1$ edges randomly to v and v' respectively (except s, t, v, v')

Node bisection



1. Choose a node v
2. Divide d neighbors of v into $\{s_1, \ldots, s_{d/2}\}$, $\{t_1, \ldots, t_{d/2}\}$
3. Cut edges v-t_1 $\cdots$ v-$t_{d/2}$
4. Add bisected node v' and a edge v-v'
5. Add edges v'-t_1 $\cdots$ v'-$t_{d/2}$
6. Add $d/2$ -1 edges randomly to v and v' respectively (except s, t, v, v')

Node bisection

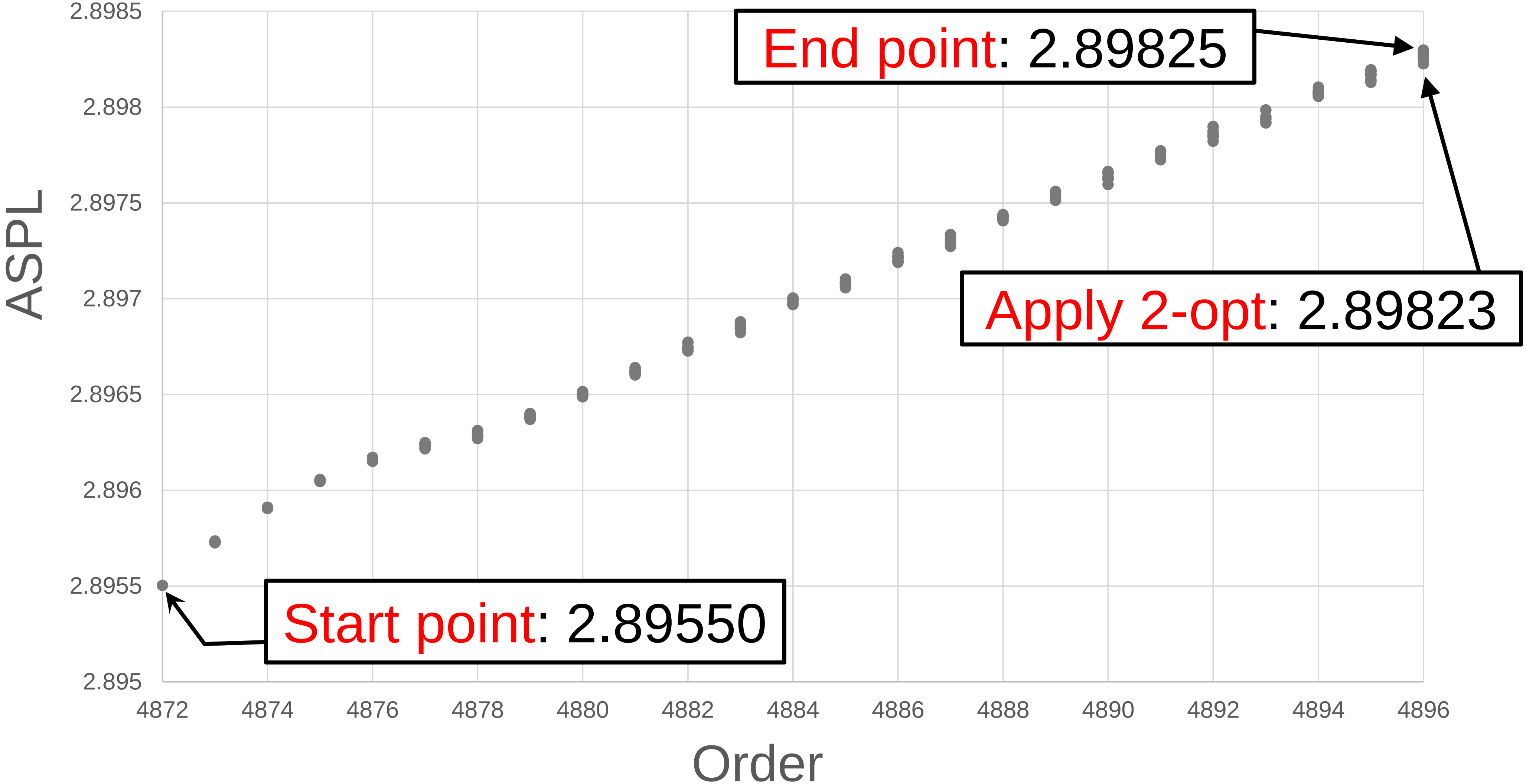


1. Choose a node v
2. Divide d neighbors of v into $\{s_1, \dots, s_{d/2}\}$, $\{t_1, \dots, t_{d/2}\}$
3. Cut edges v-t_1 $\cdots$ v-$t_{d/2}$
4. Add bisected node v' and a edge v-v'
5. Add edges v'-t_1 $\cdots$ v'-$t_{d/2}$
6. Add $d/2 - 1$ edges randomly to v and v' respectively (except s, t, v, v')

Node bisection

Target: 4896 nodes, degree 24

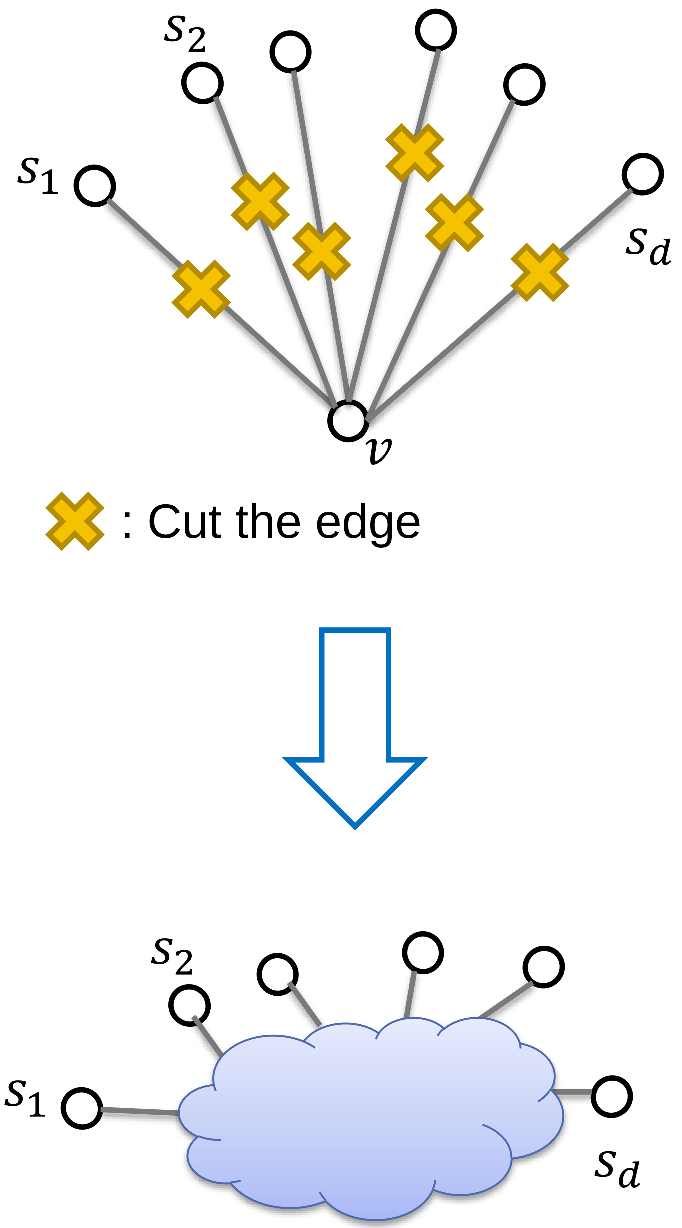
4872 nodes (Cayley graph) + 24 nodes



2. Cayley graph as a base

Node removing method

Node removing (n nodes)

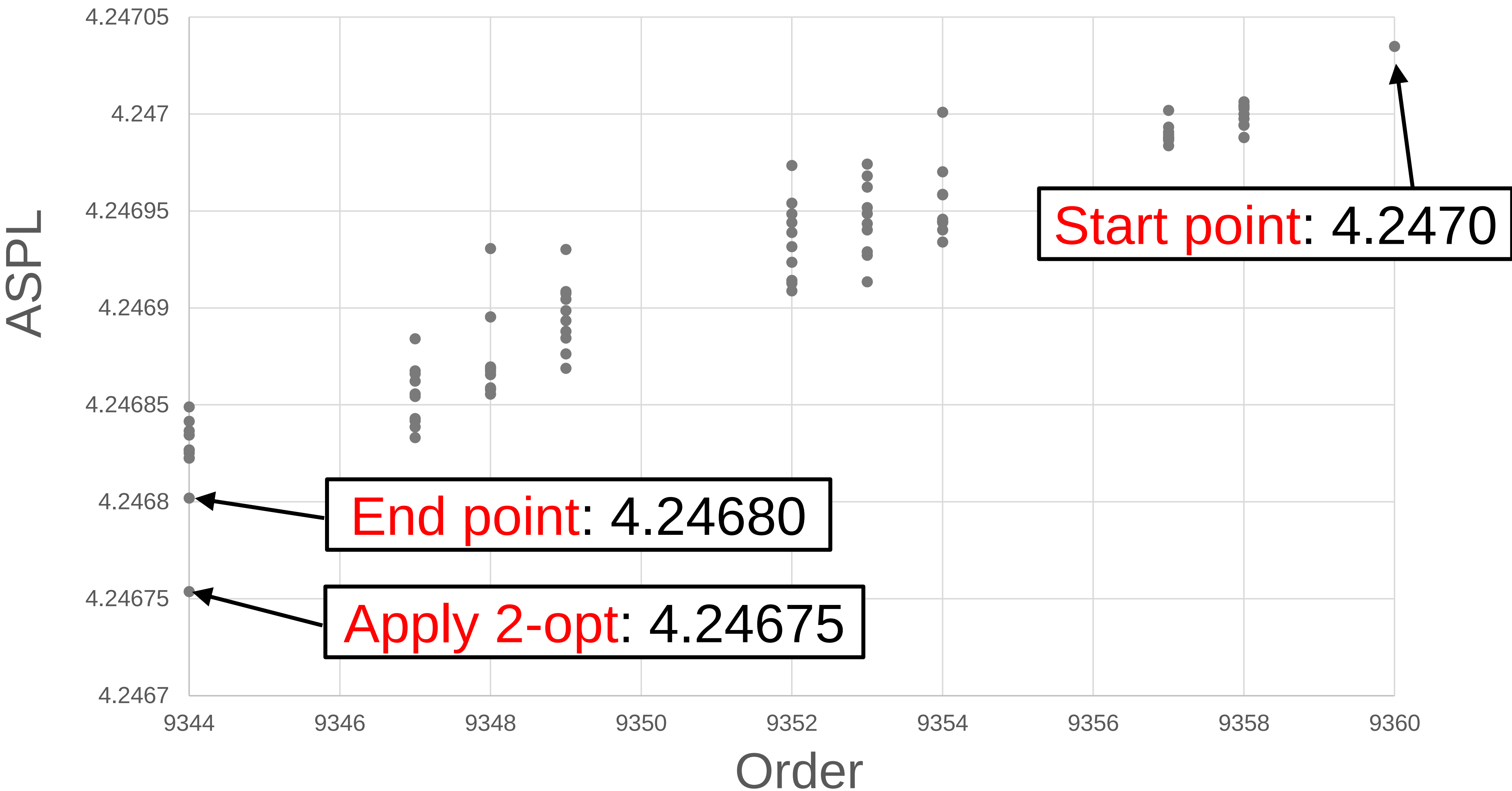


1. Choose a node v randomly
2. Cut edges v-s_1 $\cdots$ v-s_d
3. Remove a node v
4. Add edges randomly
5. Repeat 1.-4. n times

Node removing

Target: 9344 nodes, degree 10

9360 nodes (Cayley graph) – 16 nodes



Pros and Cons

Pros and Cons

- Brown’s construction
  - Small diameter $(k = 2)$
  - It may be useful for graphs only $k = 3$
- Cayley graphs
  - Fast diameter and ASPL computation
  - Can change order/degree by parameters
  - Need technique to find a better one

Conclusion

Conclusion

- Our results

  We won General graph widest improvement and Grid graph deepest improvement

- Approach

  1. Brown's construction
     - Uniting Two $B(25)$ and add nodes
  2. Cayley graph as a base
     - Node removing method
     - Node bisection method